AUTOMORPHIC FORMS FOR ELLIPTIC FUNCTION FIELDS

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ABSTRACT. Let F be the function field of an elliptic curve X over $\mathbb{F}_q$. In this paper, we calculate explicit formulas for unramified Hecke operators acting on automorphic forms over F. We determine these formulas in the language of the graph of an Hecke operator, for which we use its interpretation in terms of $\mathbb{P}^1$-bundles on X. This allows a purely geometric approach, which involves, amongst others, a classification of the $\mathbb{P}^1$-bundles on X.

We apply the computed formulas to calculate the dimension of the space of unramified cusp forms and the support of a cusp form. We show that a cuspidal Hecke eigenform does not vanish in the trivial $\mathbb{P}^1$-bundle. Further, we determine the space of unramified F'-toroidal automorphic forms where F' is the quadratic constant field extension of F. It does not contain non-trivial cusp forms. An investigation of zeros of certain Hecke L-series leads to the conclusion that the space of unramified toroidal automorphic forms is spanned by the Eisenstein series $E(\,\cdot\,,s)$ where $s+1/2$ is a zero of the zeta function of X—with one possible exception in the case that q is even and the class number h equals $q+1$.

CONTENTS

INTRODUCTION

The space of automorphic forms over $\mathbb{Q}$ is an extensively studied object and many explicit results about its structure are known, for instance, one knows the dimensions of certain subspaces like spaces of cusp forms with a fixed weight and ramification. The function field analog of the rational numbers are rational function fields $\mathbb{F}_q[T]$. The space of automorphic forms is also in these cases well-studied. A major tool of investigation are Hecke operators, for which explicit formulas are available for both $\mathbb{Q}$ and $\mathbb{F}_q[T]$. The action of Hecke operators is less known for

other global fields than $\mathbb{Q}$ and $\mathbb{F}_q[T]$. In particular, explicit formulas for the action of Hecke operators over function fields of genus 1 or higher are not available in literature—with one exception, which are elliptic function fields with odd class number. We explain below how to extract these explicit formulas from results in literature and why this method works only for odd class number. We will see that the only satisfying cases are elliptic function fields with class number 1, which includes, up to isomorphism, only three fields (cf. Example 4.1).

In this paper we shall work out explicit equations for Hecke operators over any elliptic function field. This will be done in terms of the graph of an Hecke operator, which is a tool introduced exactly for this purpose (cf. [8]). We extract formulas from these graphs and employ them to calculate the space of cusp forms, the space of toroidal automorphic forms as well as some other spaces. Note that in this exposition as well as throughout the paper, we will restrict ourselves to unramified automorphic forms and unramified Hecke operators exclusively, and we agree to suppress the attribute "unramified" from our terminology.

We continue with explaining what can be deduced from literature about Hecke operators for elliptic function fields. Let the F be the function field of an elliptic curve X over a finite field $\mathbb{F}_q$. Let x be a place of F. We denote by F_x the completion of F at x, by $\mathcal{O}_x$ its integers, by $\pi_x \in \mathcal{O}_x$ a uniformizer and by q_x the cardinality of the residue field $\mathcal{O}_x \,/\, (\pi_x) \simeq \mathbb{F}_{q_x}$. The Bruhat-Tits tree $\mathcal{T}_x$ of F_x is a graph with vertex set $\mathrm{PGL}_2(F_x) \,/\, \mathrm{PGL}_2(\mathcal{O}_x)$. There is an edge between two cosets $[g]$ and $[g']$ if and only if $[g']$ contains $g\left(\begin{smallmatrix} 1 & \\ & \pi_x \end{smallmatrix}\right)$ or $g\left(\begin{smallmatrix} \pi_x & b \\ & 1 \end{smallmatrix}\right)$ for some $b \in \mathbb{F}_{q_x}$. Note that this condition is symmetric in g and g', so $\mathcal{T}_x$ is a geometric graph. In fact, $\mathcal{T}_x$ is a (q_x+1)-regular tree.

We define the action of the local Hecke operator T_x on the space of complex valued functions f on the vertices of $\mathcal{T}_x$ by the formula

$$T_x(f)(v) \quad = \quad \sum_{v' \text{ adjacent to } v} f(v') \; .$$

Let $\mathcal{O}_F^x \subset F$ be the Dedekind ring of all elements $a \in F$ with $|a|_y \leq 1$ for all places $y \neq x$. Then $\Gamma = \mathrm{PGL}_2(\mathcal{O}_F^x)$ acts on $\mathcal{T}_x$ by left multiplication, which induces an action of Γ on the functions on $\mathrm{Vert}\ \mathcal{T}_x$. This action commutes with the action of T_x.

If the class number h of F as well as the degree of x is odd, then the strong approximation property of SL_2 implies that the inclusion of $\mathrm{PGL}_2(F_x)$ into $\mathrm{PGL}_2(\mathbb{A})$ induces a bijection

$$\Gamma \setminus \mathrm{Vert}\ \mathcal{T}_x \quad \stackrel{\sim}{\longrightarrow} \quad \mathrm{PGL}_2(F) \setminus \mathrm{PGL}_2(\mathbb{A}) \,/\, \mathrm{PGL}_2(\mathcal{O}_{\mathbb{A}})$$

where $\mathcal{O}_{\mathbb{A}}$ is the maximal compact subring of the adeles $\mathbb{A}$ of F (cf. [8, Prop. 3.8]. We denote the quotient on the right hand side by $\mathcal{X}$. An automorphic form is a function f on $\mathcal{X}$, or, equivalently, on $\Gamma \setminus \mathrm{Vert}\ \mathcal{T}_x$, which satisfies, as a (Γ-invariant) function on $\mathrm{Vert}\ \mathcal{T}_x$, that $\{T_x^i(f)\}_{i \geq 0}$ spans a finite-dimensional complex vector space. Note that the local Hecke operator T_x corresponds to a (global) Hecke operator Φ_x. To be more precise, the bijection $\Gamma \setminus \mathrm{Vert}\ \mathcal{T}_x \to \mathcal{X}$ induces an isomorphism between the function spaces on these sets, which is equivariant with respect to the operators T_x and Φ_x.

Thus we can regard f as a function on the quotient $\Gamma \setminus \mathrm{Vert}\ \mathcal{T}_x$. Shuzo Takahashi calculates this quotient for places x of degree 1 in [13]. This means that he describes representatives in $\mathrm{PGL}_2(F_x)$ for the double quotient $\Gamma \setminus \mathrm{PGL}_2(F_x) \,/\, \mathrm{PGL}_2(\mathcal{O}_x)$. The full subgraph of the classes of these representatives in $\mathrm{Vert}\ \mathcal{T}_x$ is a tree, and this tree is isomorphic to the quotient graph $\Gamma \setminus \mathcal{T}_x$.

We illustrate the quotient graph together with some matrix representatives of vertices in Figure 1. The element b varies through all elements in $\mathbb{F}_q$. Whether $\left(\begin{smallmatrix}\pi_x^2 & b\\ & 1\end{smallmatrix}\right)$ represents a vertex to the right or to the left of $\left(\begin{smallmatrix}\pi_x & 1\\ & 1\end{smallmatrix}\right)$ depends on whether the Weierstrass polynomial $P(\underline{X},\underline{Y})$ for X has a root in $\underline{X}$ for $\underline{Y}=b$ or not.

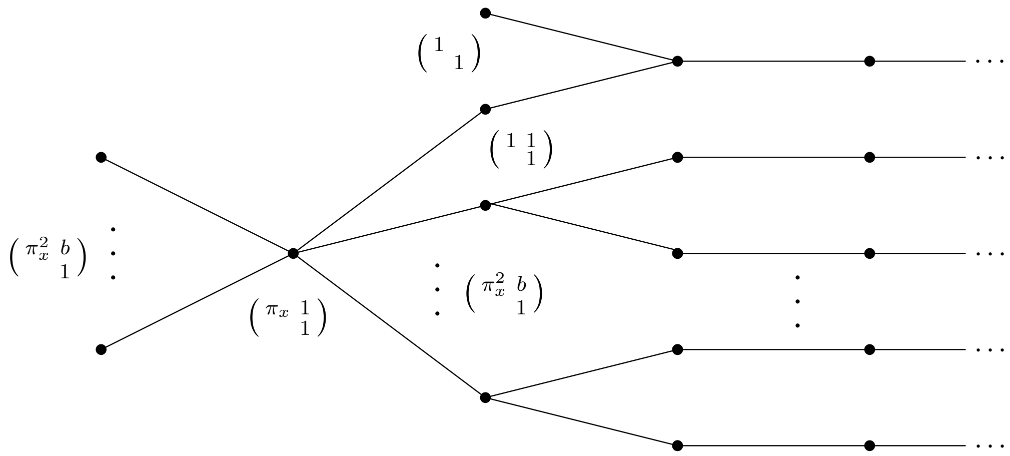

FIGURE 1. The quotient graph $\Gamma\setminus\mathcal{T}_x$

This quotient graph, in turn, is isomorphic to the graph of Φ_x (when weights are suppressed), which is illustrated in Figure 4 (see section 4). Takahashi further determines the stabilizers of Γ acting $\mathrm{Vert}\ \mathcal{T}_x$, which allows to compute the action of T_x on a function on $\Gamma\setminus\mathrm{Vert}\ \mathcal{T}_x\simeq\mathcal{X}$ by the formula

$$T_x(f)(v) \quad = \sum_{\substack{\text{edges } e \text{ with origin } v\\ \text{and terminus } v'}} [\mathrm{Stab}_\Gamma(v):\mathrm{Stab}_\Gamma(e)]\cdot f(v')\ .$$

An automorphic form that is an simultaneous eigenfunction for all Hecke operators is determined by the set of its eigenvalues. In the case that the class number is 1, the automorphic form is already determined by the eigenvalue with respect to Φ_x respective T_x, up to finitely many exceptions. For computations that profit from this point of view, see [2]. If the class number is odd, but not 1, then the eigenvalue with respect to T_x does not suffice to determine an automorphic form, but we need to consider the action of other Hecke operators as well. This leads to the difficulty to identify the representatives of $\mathcal{X}$ in $\mathrm{PGL}_2(F_x)$ with the representatives in $\mathrm{PGL}_2(F_y)$ where y is a different place of degree 1. From the viewpoint of Takahashi's paper, this seems to be a difficult problem. Even worse, if the class number is even, the correspondence between the local and the global situation breaks down and we are not able to draw any of the above conclusions.

For this reason, the notion of the graph of an Hecke operator was introduced in [8], which relates to automorphic forms as functions on $\mathcal{X}$ directly, without making use of the tree $\mathcal{T}_x$. The interpretation of $\mathcal{X}$ as the set of isomorphism classes of $\mathbb{P}^1$-bundles on X allows us to apply geometric methods, which prove to be very efficient. This gives access to a simultaneous consideration of all Hecke operators, for both odd and even class number. Note that we will restrict ourselves to the geometric viewpoint for the rest of this paper, in contrast to the above

exposition. More details on the relation between the geometric and the arithmetic setting can be found in [8, section 5].

The paper is organized into two parts. In part 1, we determine the graphs $\mathcal{G}_x$ of the Hecke operators Φ_x for degree 1 places x. In section 1, we recall the definition of the graphs $\mathcal{G}_x$ in the more general setting of an arbitrary global function field. We review the structure theory for $\mathcal{G}_x$ as developed in [8]. In particular, we explain that $\mathcal{G}_x$ decomposes into finitely many cusps, which are subgraphs that have a simple description, and into a finite subgraph, which is called the nucleus of $\mathcal{G}_x$. In section 2, we determine the vertices of the nucleus as a consequence of Atiyah's classification of vector bundles on elliptic curves. In section 3, we determine the edges in the nucleus by extensive use of the methods from [8]. In section 4, we illustrate examples of graphs of Hecke operators.

In part 2, we apply the knowledge about the graphs $\mathcal{G}_x$ to explicit calculations with automorphic forms. In section 5, we review the notion of an automorphic form as a function on the graph. We write out explicit formulas for an Hecke operator Φ_x acting on an eigenfunction. In section 6, we calculate the space of cusp forms. This means that we determine its dimension and the (maximal) support of a cusp form. We show that the space of cusp forms is a 0-eigenspace for every Φ_x where x is a place of odd degree. We prove further that a cuspidal Hecke eigenform does not vanish in the trivial $\mathbb{P}^1$-bundle. In section 7, we calculate the space of F'-toroidal automorphic forms where F' is the quadratic constant field extension of F. In particular, we see that this space contains co non-trivial cusp form. We show that the space of toroidal automorphic forms is generated by a single Eisenstein series $E(\,\cdot\,, s)$ where $s + 1/2$ is a zero of the zeta function of F—with a possible exception in the case that the characteristic of F is 2 and the class number h of F equals $q + 1$ where the space of toroidal automorphic forms might be 2-dimensional.

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Part 1. **Graphs of Hecke operators for elliptic function fields**

1. REMINDER ON GRAPHS OF HECKE OPERATORS

In this section, we recall the definition of the graphs $\mathcal{G}_x$ of the Hecke operators Φ_x as introduced in [8]. These graphs encode the action of certain unramified Hecke operators Φ_x which act on the space of automorphic forms for PGL_2 over a global function field F. The connection to the Hecke operators Φ_x will be explained in section 5. We concentrate in this resume on the geometric point of view. For the translation into adelic language, see section 5 in [8].

1.1. Let q be a prime power and X a smooth projective geometrically irreducible curve over $\mathbb{F}_q$ with function field F. Let $|X|$ be the set of closed points of X, which we identify with the set of places of F.

Let $X' = X \otimes \mathbb{F}_{q^2}$ be the constant field extension of X to $\mathbb{F}_{q^2}$ and $\overline{X} = X \otimes \overline{\mathbb{F}}_q$ the constant field extension of X to the algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$. For Y equal to X, X' or $\overline{X}$, we denote by $\mathrm{Pic}\, Y$ the Picard group of Y and by $\mathrm{Bun}_2\, Y$ the set of isomorphism classes of rank 2 bundles over Y. The Picard group $\mathrm{Pic}\, Y$ acts on $\mathrm{Bun}_2\, Y$ via the tensor product. We denote the quotient by $\mathbb{P}\mathrm{Bun}_2\, Y$, which is the same as the set of isomorphism classes of $\mathbb{P}^1$-bundles over Y. We

write $[\mathcal{M}] \in \mathbb{P}\mathrm{Bun}_2\, Y$ if the class $[\mathcal{M}]$ is represented by the rank 2 bundle $\mathcal{M}$, and $\mathcal{M} \sim \mathcal{M}'$ if $[\mathcal{M}] = [\mathcal{M}']$. We identify $\mathcal{M}$ with the associated locally free sheaf of rank 2.

Two exact sequences of sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}'_1 \to 0 \qquad \text{and} \qquad 0 \to \mathcal{F}_2 \to \mathcal{F} \to \mathcal{F}'_2 \to 0 \ ,$$

are *isomorphic with fixed* $\mathcal{F}$ if there are isomorphisms $\mathcal{F}_1 \to \mathcal{F}_2$ and $\mathcal{F}'_1 \to \mathcal{F}'_2$ such that

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_1 \arrow[r] \arrow[d, "\simeq"] & \mathcal{F} \arrow[r] \arrow[d, equal] & \mathcal{F}'_1 \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathcal{F}_2 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}'_2 \arrow[r] & 0 \end{tikzcd}$$

commutes.

Fix a place x. Let $\mathcal{K}_x$ be the torsion sheaf that is supported at x and has stalk κ_x at x, where κ_x is the residue field at x. Fix a representative $\mathcal{M}$ of $[\mathcal{M}] \in \mathbb{P}\mathrm{Bun}_2\, X$. We define $m_x([\mathcal{M}], [\mathcal{M}'])$ as the number of isomorphism classes of exact sequences

$$0 \longrightarrow \mathcal{M}'' \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}_x \longrightarrow 0$$

with fixed $\mathcal{M}$ and with $\mathcal{M}'' \sim \mathcal{M}'$. This number is independent of the choice of representative $\mathcal{M}$ (cf. [8, par. 5.4]).

For a $\mathbb{P}^1$-bundle $v \in \mathbb{P}\mathrm{Bun}_2\, X$ we define

$$\mathcal{U}_x(v) \ = \ \{(v, v', m) \mid m = m_x(v, v') \neq 0\} \ ,$$

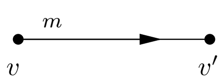
and call the occurring v' the Φ_x*-neighbours of* v, and $m_x(v, v')$ their *multiplicity*.

We define the graph $\mathcal{G}_x$ by

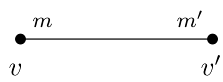
$$\begin{aligned} \mathrm{Vert}\ \mathcal{G}_x \ &= \ \mathbb{P}\mathrm{Bun}_2\, X \quad \text{and} \\ \mathrm{Edge}\ \mathcal{G}_x \ &= \ \coprod_{v \in \mathbb{P}\mathrm{Bun}_2\, X} \mathcal{U}_x(v) \end{aligned}$$

where an edges from v to v' comes together with a weight $m = m_x(v, v')$.

We list some facts about the graphs $\mathcal{G}_x$. By definition, the weight of an edge is a positive integer, and there is at most one edge between two vertices. Every edge (v, v', m) has an inverse edge, i.e. there is an edge (v', v, m') in $\mathcal{G}_x$ for some positive integer m', which in general differs from m (cf. [8, par. 3.2]). The weights of all edges (v, v', m) with origin v sum up to $q_x + 1$ where $q_x = q^{\deg x}$ is the cardinality of κ_x (cf. [8, Prop. 2.3]). In particular, $\mathcal{G}_x$ is a locally finite graph. We illustrate a single edge (v, v', m) by



and a pair of inverse edges (v, v', m) and (v', v, m') by



(note that in all examples of the present paper $v \neq v'$ for an edge (v, v', m) of $\mathcal{G}_x$). We encourage the reader to have a glance at the examples in section 4.

1.2. More details on the facts listed in this paragraph can be found in sections 6 and 7 of [8]. A line subbundle of a rank 2 bundle $\mathcal{M}$ is a morphism $\mathcal{L} \to \mathcal{M}$ where $\mathcal{L}$ is a line bundle such that the quotient $\mathcal{M} \,/\, \mathcal{L}$ is torsion free and thus also a line bundle. We define $\delta(\mathcal{L}, \mathcal{M})$ as $2 \deg \mathcal{L} - \deg \mathcal{M}$ and $\delta(\mathcal{M})$ as the supremum of $\delta(\mathcal{L}, \mathcal{M})$ over all subbundles $\mathcal{L} \to \mathcal{M}$. If $\mathcal{M} \sim \mathcal{M}'$, then $\delta(\mathcal{M}) = \delta(\mathcal{M}')$, so $\delta([\mathcal{M}]) = \delta(\mathcal{M})$ is well-defined for $[\mathcal{M}] \in \mathbb{P}\mathrm{Bun}_2 X$. For all rank 2 bundles $\mathcal{M}$ there is a line subbundle $\mathcal{L} \to \mathcal{M}$ such that $\delta(\mathcal{M}) = \delta(\mathcal{L}, \mathcal{M})$. We call such a line subbundle a *maximal subbundle of* $\mathcal{M}$. For every $v \in \mathrm{Vert}\ \mathcal{G}_x = \mathbb{P}\mathrm{Bun}_2 X$, we have $\delta(v) \geq -2g$ where g is the genus of X.

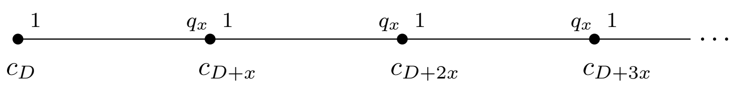
The set $\mathbb{P}\mathrm{Bun}_2 X$ is the disjoint union of the following three subsets: the set $\mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ of classes that are represented by rank 2 bundles that decompose into a direct sum of two line bundles, the set $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$ of classes that are represented by indecomposable rank 2 bundles that decompose over X' into the sum of two line bundles and the set $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$ of classes that are represented by geometrically indecomposable rank 2 bundles.

Classes in $\mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ can be represented by a rank 2 bundle of the form $\mathcal{O}_X \oplus \mathcal{L}_D$ where $\mathcal{O}_X$ is the structure sheaf of X and $\mathcal{L}_D$ is the line bundle associated to the divisor class D in the divisor class group $\mathrm{Cl}\, X$ (cf. [6, Prop. II.6.13]). We denote the corresponding class in $\mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ by c_D. We have that $c_D = c_{D'}$ if and only if $D = D'$ or $D = -D'$ in $\mathrm{Cl}\, X$ (cf. [8, Prop. 6.3]).

Classes in $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$ are represented by the trace of a line bundle $\mathcal{L}$ over X'. A line bundle over X' corresponds to a divisor class $D \in \mathrm{Cl}\, X'$, and we denote the class in $\mathbb{P}\mathrm{Bun}_2 X$ represented by the trace of $\mathcal{L}_D$ by t_D. Note that $t_D = c_0$ if and only if $D \in \mathrm{Cl}\, X \subset \mathrm{Cl}\, X'$, and otherwise $t_D \in \mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$. We have $t_D = t_{D'}$ if and only if $D = D'$ or $D = -D'$ in $\mathrm{Cl}\, X' \,/\, \mathrm{Cl}\, X$ (cf. [8, Prop. 6.4]). The integer $\delta(v)$ is even and negative for $v \in \mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$.

The set $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$ depends heavier on the arithmetic of the given curve X. For $v \in \mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$, we have that $-2g \leq \delta(v) \leq 2g - 2$. Consequently, $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$ is empty if the genus of X is 0. For genus 1, we will determine $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$ in Theorem 2.4.

1.3. We recall the definitions of the nucleus and the cusps of $\mathcal{G}_x$. More details on the following can be found in section 8 of [8]. Let $m_X = \max\{0, 2g-2\}$. The *nucleus* $\mathcal{N}_x$ *of* $\mathcal{G}_x$ is the full subgraph of $\mathcal{G}_x$ whose vertex set consists of all those $v \in \mathrm{Vert}\ \mathcal{G}_x$ with $\delta(v) \leq m_X + \deg x$. For every $D \in \mathrm{Cl}\, X$, the *cusp* $\mathcal{C}_x(D)$ is defined as the full subgraph of $\mathcal{G}_x$ whose vertex set consists of all vertices of the form $c_{D'}$ with $D' \equiv D \pmod{\langle x \rangle}$ and $\deg D' > m_X$. In particular, a cusp depends only on the class $[D] \in \mathrm{Cl}\, X / \langle x \rangle$. These classes are represented by $D \in \mathrm{Cl}\, X$ with $m_X < \deg D \leq m_X + \deg x$; consequently there are $h \deg x$ cusps where $h = \#\, \mathrm{Cl}^0 X$ is the class number of F. If D is such a representative, then the cusp $\mathcal{C}_x(D)$ looks like



The graph $\mathcal{G}_x$ is the union of the nucleus and the cusps. The union of edges is disjoint. The vertex sets of different cusps are disjoint and the intersection of $\mathrm{Vert}\ \mathcal{N}_x$ with $\mathrm{Vert}\ \mathcal{C}_x(D)$ equals $\{c_D\}$ if $m_X < \deg D \leq m_X + \deg x$. Note that both $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$ and $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$ are contained in the vertex set of $\mathcal{N}_x$.

The connected components of $\mathcal{G}_x$ stay in bijection with the 2-torsion elements of $\mathrm{Pic}\, X$. In particular, $\mathcal{G}_x$ is connected if and only if the class number of F is odd.

2. VERTICES

In this section, we determine the set $\mathbb{P}\mathrm{Bun}_2 X$ of all isomorphism classes of $\mathbb{P}^1$-bundles for a curve X of genus 1. In paragraph 1.2, we already described the subsets $\mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ and $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$. This reduces the problem to classifying the classes in $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$.

2.1. From now on, let X be a curve of genus 1 over $\mathbb{F}_q$ with function field F, $\mathrm{Cl}\, X$ the divisor class group and h the class number. The canonical sheaf ω_X is isomorphic to the structure sheaf $\mathcal{O}_X$. The map $X(\mathbb{F}_q) \to \mathrm{Cl}^1 X$, which is defined by considering an $\mathbb{F}_q$-rational point as a prime divisor of degree 1, is a bijection. We identify these sets. The choice of an $x_0 \in X(\mathbb{F}_q)$ defines the bijection

$$\begin{array}{ccc} X(\mathbb{F}_q) & \xrightarrow{\sim} & \mathrm{Cl}^0 X \\ x & \longmapsto & x - x_0 \end{array} .$$

So $X(\mathbb{F}_q)$ inherits a group structure and X becomes an elliptic curve.

2.2. The Riemann-Roch theorem reduces to $\dim_{\mathbb{F}_q}\Gamma(\mathcal{L}) - \dim_{\mathbb{F}_q}\Gamma(\mathcal{L}^{-1}) = \deg \mathcal{L}$. Since $\Gamma(\mathcal{L})$ is non-zero if and only if $\mathcal{L}$ is associated to an effective divisor (cf. [6, Prop. II.7.7(a)]), we obtain:

$$\dim_{\mathbb{F}_q}\Gamma(\mathcal{L}) \;=\; \begin{cases} 0 & \text{if } \deg\mathcal{L} \leq 0 \text{ and } \mathcal{L} \not\simeq \mathcal{O}_X, \\ 1 & \text{if } \mathcal{L} \simeq \mathcal{O}_X, \\ \deg\mathcal{L} & \text{if } \deg\mathcal{L} > 0. \end{cases}$$

By Serre duality, $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \simeq \Gamma(\mathcal{O}_X)$ is one-dimensional. The multiplicative group $\mathbb{F}_q^\times$ acts on $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$, and this action preserves the isomorphism type of the rank 2 bundle $\mathcal{M}$ that is determined by an extension of $\mathcal{O}_X$ by itself (cf. [8, par. 7.3]). Consequently, there is only one rank 2 bundle $\mathcal{M}_0$, up to isomorphism, that is a non-trivial extension of $\mathcal{O}_X$ by itself. Since $\delta(\mathcal{O}_X, \mathcal{M}_0) = 0$, [8, Lemma 7.6] implies that $\delta(\mathcal{M}_0) = 0$, and since $\mathcal{M}_0 \not\simeq \mathcal{O}_X \oplus \mathcal{O}_X$, the vector bundle $\mathcal{M}_0$ is indecomposable. Since for $v \in \mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$, we have $\delta(v) < 0$, it follows that $[\mathcal{M}_0] \in \mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$. We denote this class by s_0.

Let x be a place of degree 1 and let $\mathcal{L}_x$ denote the line bundle associated to the divisor class $[x] \in \mathrm{Cl}\, X$. The $\mathbb{F}_q$-vector space $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{L}_x) \simeq \mathrm{Hom}(\mathcal{O}_X, \mathcal{L}_x) \simeq \Gamma(\mathcal{L}_x)$ is also one-dimensional, and the non-trivial extensions define a rank 2 bundle $\mathcal{M}_x$. In this case, $\delta(\mathcal{M}_x) = \delta(\mathcal{O}_X, \mathcal{M}_x) = \deg \mathcal{O}_X - \deg \mathcal{L}_x = -1$ and $\mathcal{O}_X \to \mathcal{M}_x$ is a maximal subbundle. Indeed, if there was a subbundle $\mathcal{L} \to \mathcal{M}_x$ of positive degree, [8, Lemma 7.6] would imply that $\deg \mathcal{L} = 1$ and $\mathcal{M}_x \simeq \mathcal{O}_X \oplus \mathcal{L}$. But $\mathcal{L} \simeq \det(\mathcal{O}_X \oplus \mathcal{L}) \simeq \det \mathcal{M}_x \simeq \mathcal{L}_x$, thus we contradict the assumption that $\mathcal{M}_x$ is a non-trivial extension of $\mathcal{L}_x$ by $\mathcal{O}_X$. By the considerations of paragraph 1.2 on the values of δ we know that $[\mathcal{M}_x] \in \mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$. We denote this class by s_x.

2.3. **Remark.** The notations for the vector bundles $\mathcal{L}_x$ and $\mathcal{M}_x$ of the previous paragraph is the same as the notation for the stalk of some vector bundles $\mathcal{L}$ respective $\mathcal{M}$ at x. To avoid confusion, we will reserve the notations $\mathcal{L}_x$ and $\mathcal{M}_x$ strictly for the vector bundles as defined in the last paragraph throughout the whole paper.

2.4. **Theorem.**

$$\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X \;=\; \left\{\, s_x \;\middle|\; x \in \mathrm{Cl}^1 X \,\right\} \;\amalg\; \left\{\, s_0 \,\right\} ,$$

and $s_x = s_y$ *if and only if* $(x - y) \in 2\,\mathrm{Cl}^0 X$.

Proof. Let Y denote one of X, $X' = X \otimes \overline{\mathbb{F}}_{q^2}$ or $\overline{X} = X \otimes \overline{\mathbb{F}}_q$. Let $\mathcal{B}_n^d(Y)$ be the set of isomorphism classes of geometrically indecomposable rank n bundles over Y that have degree d. We have inclusions $\mathcal{B}_n^d(X) \subset \mathcal{B}_n^d(X') \subset \mathcal{B}_n^d(\overline{X})$ (cf. [8, par. 6.1]). For a rank 1 bundle $\mathcal{L}$ over Y, the map

$$\begin{array}{ccc} \mathcal{B}_n^d(Y) & \longrightarrow & \mathcal{B}_n^{d+rn}(Y) \\ \mathcal{M} & \longmapsto & \mathcal{M} \otimes \mathcal{L}^r \end{array}$$

defines a bijection of sets for every $d, r \in \mathbb{Z}$ and $n \geq 1$. We have to determine the orbits of $\operatorname{Pic}^0 X$ in $\mathcal{B}_2^0(X)$ and $\mathcal{B}_2^1(X)$ to verify the theorem. We already know that $\mathcal{M}_0 \in \mathcal{B}_2^0(X)$ and $\mathcal{M}_x \in \mathcal{B}_2^1(X)$ for all $x \in \operatorname{Cl}^1 X$.

For the case $d = 0$, we use the following result of Atiyah.

2.5. **Theorem** ([1, Thm. 5 (ii)]). *For all $\mathcal{M}, \mathcal{M}' \in \mathcal{B}_n^0(\overline{X})$, there exists a unique $\mathcal{L} \in \operatorname{Pic}^0 \overline{X}$ such that $\mathcal{M} \simeq \mathcal{M}' \otimes \mathcal{L}$.*

This implies that for every $\mathcal{M} \in \mathcal{B}_2^0(X)$, there exists a unique $\mathcal{L} \in \operatorname{Pic}^0 \overline{X}$ such that $\mathcal{M} \simeq \mathcal{M}_0 \otimes \mathcal{L}$. The action of $\operatorname{Pic}^0 \overline{X}$ and $\operatorname{Gal}(\overline{\mathbb{F}}_q \,/\, \mathbb{F}_q)$ on vector bundles over $\overline{X}$ commute, and thus for every $\sigma \in \operatorname{Gal}(\overline{\mathbb{F}}_q \,/\, \mathbb{F}_q)$,

$$\mathcal{M}_0 \otimes \mathcal{L}^\sigma \;\simeq\; (\mathcal{M}_0 \otimes \mathcal{L})^\sigma \;\simeq\; \mathcal{M}^\sigma \;\simeq\; \mathcal{M} \;\simeq\; \mathcal{M}_0 \otimes \mathcal{L} \,.$$

By uniqueness, $\mathcal{L}^\sigma \simeq \mathcal{L}$; thus $\mathcal{L} \in \operatorname{Pic}^0 X$. Hence $[\mathcal{M}] = s_0 \in \mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$.

For $d = 1$, we restate Atiyah's classification of indecomposable vector bundles over $\overline{X}$.

2.6. **Theorem** ([1, Thm. 7]). *There are bijections $\varphi_n^d : \mathcal{B}_n^d(\overline{X}) \to \operatorname{Pic}^0(\overline{X})$ such that the diagrams*

$$\begin{array}{ccc} \mathcal{B}_n^d(\overline{X}) & \xrightarrow{\varphi_n^d} & \operatorname{Pic}^0(\overline{X}) \\ \Big\downarrow{\scriptstyle \det} & & \Big\downarrow{\scriptstyle (n,d)} \\ \mathcal{B}_1^d(\overline{X}) & \xrightarrow{\varphi_1^d} & \operatorname{Pic}^0(\overline{X}) \end{array}$$

commute for all $d \in \mathbb{Z}$ and $n \geq 1$. Here (n, d) denotes multiplication with the greatest common divisor of n and d.

This means that $\det : \mathcal{B}_2^1(\overline{X}) \to \mathcal{B}_1^1(\overline{X})$ is a bijection, and consequently the restriction $\det : \mathcal{B}_2^1(X) \to \mathcal{B}_1^1(X)$ is still injective. Because every element of $\mathcal{B}_1^1(X)$ is of the form $\mathcal{L}_x$ for some place x of degree 1 and because $\det(\mathcal{M}_x) \simeq \mathcal{L}_x \in \mathcal{B}_1^1(X)$, we obtain that $\mathcal{B}_2^1(X) = \{\mathcal{M}_x | x \in \operatorname{Cl}^1 X\}$.

By the injectivity of the determinant map, $\mathcal{M}_x \simeq \mathcal{M}_y \otimes \mathcal{L}$ for some $\mathcal{L} \in \operatorname{Pic}^0 X$ if and only if $\det \mathcal{M}_x \simeq \det(\mathcal{M}_y \otimes \mathcal{L}) \simeq (\det \mathcal{M}_y) \otimes \mathcal{L}^2$, or, equivalently, $(x - y) \in 2 \operatorname{Cl}^0 X$. This proves Theorem 2.4. □

2.7. **Remark.** Theorem 2.4 shows that non-isomorphic $\mathbb{P}^1$-bundles that are geometrically indecomposable may become isomorphic after extension of the base field—in contrast to the opposite result for $\mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ and $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$ ([8, Lemma 6.5]). Namely, if $x - y \notin 2 \operatorname{Cl}^0 X$, then s_x and s_y are not isomorphic. But there is a finite constant extension $Y \to X$ such that $x - y \in 2 \operatorname{Cl}^0 Y$, since geometrically the class group of an elliptic curve is divisible. Thus s_x and s_y become isomorphic over Y. For a concrete example, consider $X = X_6$, and $Y = X_6'$ as in paragraph 4.4.

2.8. **Corollary.** *If a rank* 2 *bundle* $\mathcal{M}$ *has* $\delta(\mathcal{M}) = -1$ *and* $\det \mathcal{M} \simeq \mathcal{L}_x$*, then* $\mathcal{M}$ *represents* s_x*.*

Proof. A rank 2 bundle $\mathcal{M}$ with $\delta(\mathcal{M}) = -1$ must be geometrically indecomposable. The corollary follows from the fact that every element of $\mathcal{B}_2^1(X)$ is characterised by its determinant. □

2.9. **Corollary.** *Let* $x \in X(\mathbb{F}_q)$*. Then the nucleus* $\mathcal{N}_x$ *of the graph* $\mathcal{G}_x$ *consists of the vertices*

$$\operatorname{Vert} \mathcal{N}_x \;=\; \{\, t_D \,\}_{D \in \operatorname{Cl} X' - \operatorname{Cl} X} \;\amalg\; \{\, s_x \,\}_{x \in \operatorname{Cl}^1 X} \;\amalg\; \{\, s_0 \,\} \;\amalg\; \{\, c_D \,\}_{D \in \operatorname{Cl}^0 X \cup \operatorname{Cl}^1 X} \,.$$

Proof. Theorem 2.4 describes $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$, which is contained in the vertex set of the nucleus. The description of all other vertices follows from the definition of the nucleus and the classification of $\mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}} X$ and $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$ as described in paragraph 1.2. □

3. EDGES

In this section, we investigate the edges of the graphs $\mathcal{G}_x$ for degree 1 places x, i.e. $x \in X(\mathbb{F}_q)$. In Section 1, we described these graphs up to the nucleus $\mathcal{N}_x$ and in the previous section we determined the vertices of $\mathcal{N}_x$. We illustrate our knowledge in Figure 2.

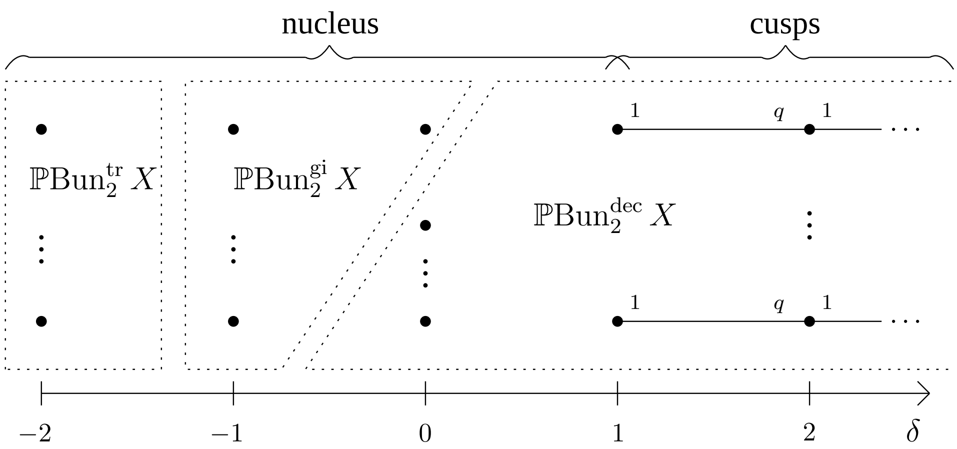


FIGURE 2. $\mathcal{G}_x$ up to a finite number of edges

Fix a place x of degree 1. The divisor classes of degree 0 can be represented by $x - z$ where z is place of degree 1 which is uniquely determined by the divisor class $x - z$. The following theorem characterizes all missing edges of $\mathcal{G}_x$.

3.1. **Theorem.** *Let x be a prime divisor of degree 1 and $h_2 = \# \operatorname{Cl}^0 X[2]$ the cardinality of the 2-torsion of the class group. Then the edges with origin in $\mathcal{N}_x$ are given by the following list:*

$$\begin{aligned}
\mathcal{U}_x(c_0) &= \{(c_0, c_x, q+1)\} \;, \\
\mathcal{U}_x(c_x) &= \{(c_x, c_{2x}, 1), (c_x, c_0, 1), (c_x, s_0, q-1)\} \;, \\
\mathcal{U}_x(c_y) &= \{(c_y, c_{y+x}, 1), (c_y, c_{y-x}, q)\} \qquad \textit{if } y \neq x, \\
\mathcal{U}_x(c_{y-x}) &= \{(c_{y-x}, c_y, 2), (c_{y-x}, s_y, q-1)\} \qquad \textit{if } y \neq x, \textit{ but } y - x \in (\operatorname{Cl} X)[2], \\
\mathcal{U}_x(c_{y-x}) &= \{(c_{y-x}, c_y, 1), (c_{y-x}, c_{2x-y}, 1), (c_{y-x}, s_y, q-1)\} \qquad \textit{if } y - x \notin (\operatorname{Cl} X)[2], \\
\mathcal{U}_x(s_0) &= \{(s_0, c_x, 1), (s_0, s_x, q)\} \;, \\
\mathcal{U}_x(t_D) &= \{(t_D, s_{x+D+\sigma D}, q+1)\} \qquad \textit{for } D \in \operatorname{Cl}^0 X' - \operatorname{Cl}^0 X \textit{ and} \\
\mathcal{U}_x(s_y) &= \{(s_y, s_0, h_2) \mid \textit{if } y \equiv x \pmod{2 \operatorname{Cl}^0 X}\} \\
&\quad \cup \left\{ (s_y, c_{z-x}, \tfrac{1}{2} h_2) \;\middle|\; \substack{\textit{if } (z-x) \in (\operatorname{Cl}^0 X)[2], \\ z \neq x \textit{ and } (z-y) \in 2 \operatorname{Cl}^0 X} \right\} \\
&\quad \cup \left\{ (s_y, c_{z-x}, h_2) \;\middle|\; \substack{\textit{if } (z-x) \notin (\operatorname{Cl}^0 X)[2] \\ \textit{and } (z-y) \in 2 \operatorname{Cl}^0 X} \right\} \\
&\quad \cup \left\{ (s_y, t_D, \tfrac{1}{2} h_2) \;\middle|\; \substack{\textit{if } D \in (\operatorname{Cl}^0 X' - \operatorname{Cl}^0 X),\, 2D \in \operatorname{Cl}^0 X \\ \textit{and } y \equiv D + \sigma D + x \pmod{2 \operatorname{Cl}^0 X}} \right\} \\
&\quad \cup \left\{ (s_y, t_D, h_2) \;\middle|\; \substack{\textit{if } D \in (\operatorname{Cl}^0 X' - \operatorname{Cl}^0 X),\, 2D \notin \operatorname{Cl}^0 X \\ \textit{and } y \equiv D + \sigma D + x \pmod{2 \operatorname{Cl}^0 X}} \right\} \qquad \textit{for } y \in \operatorname{Cl}^1 X.
\end{aligned}$$

Proof. The rest of this section is dedicated to the proof of the theorem. There are illustrations of the sets described in the theorem at the appropriate places in the proof. We draw vertices v from left to right to indicate an increasing value of $\delta(v)$. In section 4 we show illustrations of entire graphs.

We recall some results that we will use in the proof without further reference. The weights of all Φ_x-neighbours of each vertex sum up to $q+1$ ([8, Prop. 2.3]). If v and w are Φ_x-neighbours, then $\delta(w) = \delta(v) \pm 1$ ([8, Lemma 8.2]). The Φ_x-neighbours v' of a vertex $v = [\mathcal{M}]$ with $\delta(v') = \delta(v) + 1$ counted with multiplicity are in bijection with the maximal subbundles of $\mathcal{M}$ ([8, Lemma 8.4]). This bijection is given by associating to a maximal subbundle $\mathcal{L} \to \mathcal{M}$ the unique sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{K}_x \to 0$ such that $\mathcal{L} \to \mathcal{M}$ lifts to a subbundle $\mathcal{L} \to \mathcal{M}'$ (cf. [8, par. 8.3]). We call this sequence the *sequence associated to* $\mathcal{L} \to \mathcal{M}$.

We shall also need the following lemma. Let $\mathcal{J}_x$ be the kernel of $\mathcal{O}_X \to \mathcal{K}_x$, which is the inverse of the line bundle $\mathcal{L}_x$ in $\operatorname{Pic} X$.

3.2. **Lemma.** *Let $\mathcal{L} \to \mathcal{M}$ be a line subbundle and*

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}_x \longrightarrow 0$$

the associated sequence. Let $\mathcal{L}' = \mathcal{M}/\mathcal{L}$. If $\mathcal{M} \simeq \mathcal{L} \oplus \mathcal{L}'$, then $\mathcal{M}' \simeq \mathcal{L} \oplus \mathcal{L}'\mathcal{J}_x$.

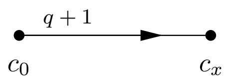
Proof. Note that $\mathcal{M}'/\mathcal{L} \simeq (\det \mathcal{M})\mathcal{J}_x\mathcal{L}^\vee \simeq \mathcal{L}'\mathcal{J}_x$. The hypothesis can be illustrated by the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{L} \arrow[r] \arrow[d, equal] & \mathcal{M}' \arrow[r] \arrow[d] & \mathcal{L}'\mathcal{J}_x \arrow[r] \arrow[dl] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{M} \arrow[r] & \mathcal{L}' \arrow[r] \arrow[l, bend right] & 0 \,.
\end{tikzcd}$$

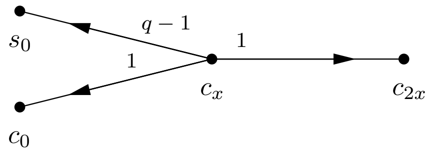
Since the composition $\mathcal{L}'\mathcal{J}_x \to \mathcal{M} \to \mathcal{K}_x$ is zero, $\mathcal{L}'\mathcal{J}_x \to \mathcal{M}$ lifts to $\mathcal{L}'\mathcal{J}_x \to \mathcal{M}'$, and the upper sequence also splits. □

We prove the theorem case by case.

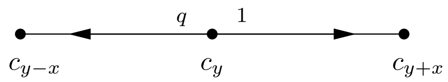
- By [8, Thm. 8.5], c_x is the only Φ_x-neighbour of c_0, which describes $\mathcal{U}_x(c_0)$ completely:



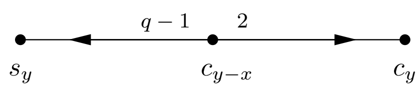
- Let $\mathcal{M} = \mathcal{L}_x \oplus \mathcal{O}_X$ represent c_x. We know from [8, Thm. 8.5] that c_{2x} is the only neighbour $\mathcal{M}'$ with $\delta(\mathcal{M}') = 2$. It has multiplicity 1 and is given by the sequence associated to $\mathcal{L}_x \to \mathcal{M}$. By Lemma 3.2, the sequence associated to $\mathcal{O}_X \to \mathcal{M}$ gives $\mathcal{O}_X \oplus \mathcal{O}_X$ as neighbour. For all other $q-1$ neighbours $\mathcal{M}'$, neither $\mathcal{L}_x \to \mathcal{M}$ nor $\mathcal{O}_X \to \mathcal{M}$ lifts to $\mathcal{M}'$, but then $\mathcal{L}_x\mathcal{J}_x \subset \mathcal{L}_x \to \mathcal{M}$ lifts to a subbundle $\mathcal{O}_X \simeq \mathcal{L}_x\mathcal{J}_x \to \mathcal{M}'$. We have that $\det \mathcal{M}' \simeq (\det \mathcal{M})\mathcal{J}_x \simeq \mathcal{O}_X$, but $\mathcal{O}_X \to \mathcal{M}'$ cannot have a complement, since otherwise $\mathcal{O}_X \to \mathcal{M}$ would lift. Thus $\mathcal{M}'$ must represent s_0. This describes $\mathcal{U}_x(c_x)$:



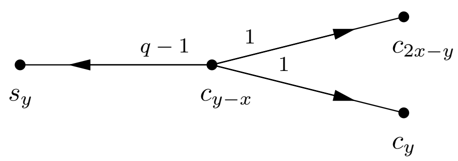
- Let $\mathcal{M} = \mathcal{L}_y \oplus \mathcal{O}_X$ represent c_y with $y \neq x$. Again, we know from [8, Thm. 8.5] that c_{y+x} is the only neighbour $\mathcal{M}'$ with $\delta(\mathcal{M}') = 2$, and it has multiplicity 1. For all other q neighbours, $\mathcal{L}_y\mathcal{J}_x \to \mathcal{M}'$ is a subbundle, and $\mathcal{M}' \,/\, \mathcal{L}_y\mathcal{J}_x \simeq \mathcal{O}_X$. But since $\mathcal{L}_y\mathcal{J}_x \not\simeq \mathcal{O}_X$, we have that $\operatorname{Ext}^1(\mathcal{L}_y\mathcal{J}_x, \mathcal{O}_X) = 0$ (paragraph 2.2), and thus $\mathcal{M}'$ decomposes. We obtain for $\mathcal{U}_x(c_y)$:



- Let $\mathcal{M} = \mathcal{L}_y \oplus \mathcal{L}_x$ represent c_{y-x} with $y \neq x$. Then the sequences associated to the two maximal subbundles $\mathcal{L}_y \to \mathcal{M}$ and $\mathcal{L}_x \to \mathcal{M}$ determine two neighbours $\mathcal{L}_y \oplus \mathcal{O}_X$ and $\mathcal{L}_y\mathcal{J}_x \oplus \mathcal{L}_x$. They both decompose by Lemma 3.2 and represent c_y and c_{2x-y}, respectively. For all other $q-1$ neighbours $\mathcal{M}'$, no maximal line bundle lifts, and thus $\delta(\mathcal{M}') = -1$. Since $\det \mathcal{M}' \simeq \mathcal{L}_y\mathcal{L}_x\mathcal{J}_x \simeq \mathcal{L}_y$, by Corollary 2.8, $\mathcal{M}'$ represents s_y. We have $c_{2x-y} = c_y$ if and only if $\mathcal{L}_x^2\mathcal{L}_y^{-1} \simeq \mathcal{L}_y$, or equivalently, $\left(\mathcal{L}_x\mathcal{L}_y^{-1}\right)^2 \simeq \mathcal{O}_X$. This means that these two neighbours are the same if and only if $x - y \in (\operatorname{Cl} X)[2]$. If this is the case, we get for $\mathcal{U}_x(c_{y-x})$:



- If $x - y \notin (\operatorname{Cl} X)[2]$, $\mathcal{U}_x(c_{y-x})$ looks like:



- Let $\mathcal{M}$ be the bundle $\mathcal{M}_0$ of paragraph 2.2, which represents s_0. Then it has a unique maximal subbundle $\mathcal{O}_X \to \mathcal{M}$ and an associated neighbour $\mathcal{M}'$ with $\delta(\mathcal{M}') = 1$, which decomposes. Because its maximal subbundle is $\mathcal{O}_X \to \mathcal{M}'$, $\det \mathcal{M}' \simeq \mathcal{J}_x$, and we recognize it as

$\mathcal{O}_X \oplus \mathcal{J}_x$. Thus $\mathcal{M}'$ represents c_x. All q other neighbours $\mathcal{M}'$ have $\delta(\mathcal{M}') = -1 = \delta(\mathcal{M}' \otimes \mathcal{L}_x)$, and $\det(\mathcal{M}' \otimes \mathcal{L}_x) \simeq \mathcal{J}_x \mathcal{L}_x^2 \simeq \mathcal{L}_x$. By Corollary 2.8, $\mathcal{M}' \otimes \mathcal{L}_x$ and thus also $\mathcal{M}'$ represent s_x, and $\mathcal{U}_x(s_0)$ is as follows:

$$\begin{tikzcd} s_x & s_0 \arrow[l, "q"'] \arrow[r, "1"] & c_x \end{tikzcd}$$

• Let $\mathcal{M}$ represent t_D for a $D \in \mathrm{Cl}^0 X' - \mathrm{Cl}^0 X$. Since $\delta(t_D) = -2$, every neighbour $\mathcal{M}'$ of $\mathcal{M}$ must have $\delta(\mathcal{M}') = -1$. It is determined by its determinant, which we can calculate by extending constants to X'. We have $\det \mathcal{M}' \simeq \mathcal{J}_x \det(\mathcal{L}_D \oplus \mathcal{L}_{\sigma D}) \simeq \mathcal{J}_x \mathcal{L}_D \mathcal{L}_{\sigma D}$. Because $-x+D+\sigma D \equiv x+D+\sigma D \pmod{2\,\mathrm{Cl}\,X}$, Corollary 2.8 implies that $\mathcal{M}'$ represents $s_{x+D+\sigma D}$. We obtain for $\mathcal{U}_x(t_D)$:

$$\begin{tikzcd} t_D \arrow[r, "q+1"] & s_{x+D+\sigma D} \end{tikzcd}$$

• The most subtle part is to determine the neighbours of s_y for $y \in \mathrm{Cl}^1 X$. We choose $\mathcal{M}_y$ as representative for s_y, see paragraph 2.2, and recall that it was defined by a nontrivial element in $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{L}_y)$. Thus $\det(\mathcal{M}_y) = \mathcal{L}_y$, and $\delta(\mathcal{M}_y) = -1$. Look at an exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M}_y \longrightarrow \mathcal{K}_x \longrightarrow 0 \quad .$$

Then $\det(\mathcal{M}') \simeq (\det \mathcal{M}_y)\mathcal{J}_x \simeq \mathcal{L}_{y-x} \in \mathrm{Pic}^0 X$, and $\delta(\mathcal{M}') \in \{-2, 0\}$. By the symmetry of edges (see paragraph 1.1), s_y must also be a neighbour of $[\mathcal{M}']$. But we have already determined the neighbours of vertices v with these properties. We find that for $(z - x) \in \mathrm{Cl}^0 X - \{0\}$, c_{z-x} is a neighbour of s_y if and only if $y \equiv z \pmod{2\,\mathrm{Cl}^0 X}$, t_D with $D \in \mathrm{Cl}^0 X' - \mathrm{Cl}^0 X$ is a neighbour of s_y if and only if $y \equiv x + D + \sigma D \pmod{2\,\mathrm{Cl}^0 X}$, and s_0 is a neighbour of s_y if and only if $y \equiv x \pmod{2\,\mathrm{Cl}^0 X}$, but c_0 is never a neighbour of s_y. This shows that the theorem lists precisely the neighbours of s_y. There is still some work to be done to determine the weights. We begin with an observation.

3.3. **Lemma.** *Up to isomorphism with fixed $\mathcal{M}_y$, there is at most one exact sequence $0 \to \mathcal{M}' \to \mathcal{M}_y \to \mathcal{K}_x \to 0$ for a fixed $\mathcal{M}'$.*

Proof. Suppose there are two. We derive a contradiction as follows. If $\delta(\mathcal{M}') \neq 0$, then $\mathcal{M}'$ must be a trace of a line bundle $\mathcal{L}$ defined over X'. By extending constants to $\mathbb{F}_{q^2}$, we may assume that $\delta(\mathcal{M}') = 0$ and that there are $\mathcal{L}, \mathcal{L}' \in \mathrm{Pic}^0 X$ such that $\mathcal{M}'$ is an extension of $\mathcal{L}'$ by $\mathcal{L}$. The composition $\mathcal{L} \to \mathcal{M}' \to \mathcal{M}$ defines a maximal subbundle of $\mathcal{M}$ because $\delta(\mathcal{L}, \mathcal{M}) = -1$. We get back the inclusion $\mathcal{M}' \to \mathcal{M}$ by taking the associated sequence. Since we assume we have two different inclusions of $\mathcal{M}'$ into $\mathcal{M}$, we get two different subbundles of the form $\mathcal{L} \to \mathcal{M}$, thus an inclusion $\mathcal{L} \oplus \mathcal{L} \to \mathcal{M}$. The cokernel is a torsion sheaf of degree 1 defined over $\mathbb{F}_{q^2}$, say $\mathcal{K}_{x'}$ for a place x' of $\mathbb{F}_{q^2}F$, and we obtain an exact sequence

$$0 \longrightarrow \mathcal{L} \oplus \mathcal{L} \longrightarrow \mathcal{M}_y \longrightarrow \mathcal{K}_{x'} \longrightarrow 0 \quad ;$$

$c_0 = [\mathcal{L} \oplus \mathcal{L}]$ is thus an $\Phi_{x'}$-neighbour of $\mathcal{M}_y$. This is a contradiction as s_y is not a neighbour of c_0. □

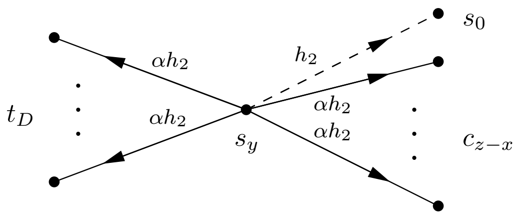
We consider a second neighbour $\mathcal{M}''$ of $\mathcal{M}_y$ that represents the same element as $\mathcal{M}'$ in $\mathbb{P}\mathrm{Bun}\, X$, i.e. $\mathcal{M}'' \simeq \mathcal{M}' \otimes \mathcal{L}_0$ for some $\mathcal{L}_0 \in \mathrm{Pic}\, X$. Since they have the same determinant,

$\mathcal{L}_0^2 \simeq \det(\mathcal{M}' \otimes \mathcal{L}_0)(\det \mathcal{M}')^{-1} \simeq (\det \mathcal{M}'')(\det \mathcal{M}')^{-1} \simeq \mathcal{O}_x$, we have $\mathcal{L}_0 \in (\operatorname{Pic} X)[2]$. On the other hand, Theorem 2.6 tells us that for $\mathcal{M}_y \in \mathcal{B}_2^1(X)$, $\mathcal{M}_y \otimes \mathcal{L}_0 \simeq \mathcal{M}_y$ if and only if $\mathcal{L}_0 \in (\operatorname{Pic} X)[2]$. Thus $(\operatorname{Pic} X)[2]$ acts on the sequences that we investigate. By Lemma 3.3, we find that the multiplicity of a neighbour $\mathcal{M}'$ of $\mathcal{M}_y$ equals the number of isomorphism classes that $\mathcal{M}' \otimes \mathcal{L}_0$ meets as $\mathcal{L}_0$ varies through $(\operatorname{Pic} X)[2] = (\operatorname{Pic}^0 X)[2]$.

We begin with the case of a neighbour $\mathcal{M}'$ that is associated to a maximal subbundle $\mathcal{L} \to \mathcal{M}_y$. Then $\delta(\mathcal{L}, \mathcal{M}') = 0$. If $\mathcal{M}' \,/\, \mathcal{L} \simeq \mathcal{L}$, the only possibility with these properties is s_0. But then $\mathcal{L} \to \mathcal{M}'$ is the only maximal subbundle, so all $\mathcal{L} \otimes \mathcal{L}_0$ with $\mathcal{L}_0 \in (\operatorname{Pic}^0 X)[2]$ have different associated sequences, and the multiplicity of s_0 is therefore $h_2 = \#(\operatorname{Pic}^0 X)[2]$.

If $\mathcal{L}' := \mathcal{M}' \,/\, \mathcal{L} \not\simeq \mathcal{L}$, then $\mathcal{M}'$ represents c_{z-x} for the divisor $(z - x) \in \operatorname{Cl}^0 X$ that satisfies $\mathcal{L}_{z-x} \simeq \mathcal{L}'\mathcal{L}^{-1}$. Since $\mathcal{L}_{y-x} \simeq \det \mathcal{M}' \simeq \mathcal{L}\mathcal{L}'$, we have $z \equiv y \pmod{2 \operatorname{Cl}^0 X}$. The rank 2 bundle $\mathcal{M}'$ has two different maximal subbundles, and it could happen that $\mathcal{M}' \simeq \mathcal{M}' \otimes \mathcal{L}_0$ for some $\mathcal{L}_0 \in (\operatorname{Pic}^0 X)[2] - \{\mathcal{O}_X\}$. This only happens if $\mathcal{L}' \simeq \mathcal{L}\mathcal{L}_0$, so $\mathcal{L}'\mathcal{L}^{-1} \in (\operatorname{Pic}^0 X)[2]$, or equivalently, $(z - x) \in (\operatorname{Cl}^0 X)[2]$. Thus the multiplicity of c_{z-x} as a neighbour of s_y is $h_2/2$ if $(z - x) \in (\operatorname{Cl}^0 X)[2] - \{0\}$ and h_2 if $(z - x) \notin (\operatorname{Cl}^0 X)[2]$.

The last case is that of $\delta(\mathcal{M}') = -2$, where $\mathcal{M}'$ is the trace of a line bundle $\mathcal{L}_D$, where $D \in \operatorname{Cl}^0 X' - \operatorname{Cl}^0 X$. If we lift the situation to X', then $\mathcal{M}' \simeq \mathcal{L}_D \oplus \mathcal{L}_{\sigma D}$, and we see as in the preceding case that $\mathcal{M}' \simeq \mathcal{M}' \otimes \mathcal{L}_0$ for some $\mathcal{L}_0 \in (\operatorname{Pic}^0 X)[2] - \{\mathcal{O}_X\}$ if and only if $D - \sigma D \in (\operatorname{Cl}^0 X)[2]$. This is equivalent to the two conditions $D - \sigma D \in \operatorname{Cl}^0 X$ and $2D - 2\sigma D = 0$, or $2D = (D - \sigma D) + (D + \sigma D) \in \operatorname{Cl}^0 X$ and $2D = \sigma(2D)$, respectively, both saying that $2D \in \operatorname{Cl}^0 X$. This finally gives for $D \in \operatorname{Cl}^0 X' - \operatorname{Cl}^0 X$ that t_D has multiplicity $h_2/2$ as neighbour of s_y if $2D \in \operatorname{Cl}^0 X$ and h_2 if $2D \notin \operatorname{Cl}^0 X$. We illustrate this below. The dashed arrow only occurs if $y - x \in 2 \operatorname{Cl}^0 X$. The indices z and D take all possible values as in the theorem, and $\alpha \in \{1/2, 1\}$ depends on the particular edge.



This completes the proof of the theorem. $\square$

4. Examples

This section provides some examples of graphs of Hecke operators.

4.1. **Example** (class number one)**.** The easiest examples are given by elliptic curves with only one rational point x. These examples can be found in the literature, cf. [2], [12, 2.4.4 and Ex. 3 of 2.4] or [13]. There are up to isomorphism three such elliptic curves: X_2 over $\mathbb{F}_2$ defined by the Weierstrass equation $\underline{Y}^2 + \underline{Y} = \underline{X}^3 + \underline{X} + 1$, X_3 over $\mathbb{F}_3$ defined by the Weierstrass equation $\underline{Y}^2 = \underline{X}^3 + 2\underline{X} + 2$ and X_4 over $\mathbb{F}_4$ defined by the Weierstrass equation $\underline{Y}^2 + \underline{Y} = \underline{X}^3 + \alpha$ with $\mathbb{F}_4 = \mathbb{F}_2(\alpha)$. Since the class number is 1, $\mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}} X_q = \{c_{nx}\}_{n \geq 0}$ and $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X_q = \{s_0, s_x\}$ for $q \in \{2, 3, 4\}$. One calculates that $\operatorname{Cl}^0(X_2 \otimes \mathbb{F}_4) \simeq \mathbb{Z}/5\mathbb{Z}$, $\operatorname{Cl}^0(X_3 \otimes \mathbb{F}_9) \simeq \mathbb{Z}/7\mathbb{Z}$ and $\operatorname{Cl}^0(X_4 \otimes \mathbb{F}_{16}) \simeq \mathbb{Z}/9\mathbb{Z}$, thus $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X_q$ has q different elements $t_1, \dots, t_q$ for $q \in \{2, 3, 4\}$. We obtain Figure 3.

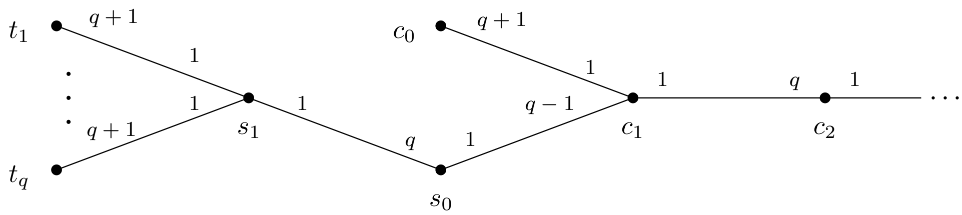


FIGURE 3. $\mathcal{G}_x$ for the unique degree one place x of the elliptic curves X_q for $q = 2, 3, 4$

4.2. **Example** (odd class number)**.** More generally, let the class number h be odd. Let x a place of degree 1. Then $\mathcal{G}_x$ has only one component. We write $\{x, z_2, \dots, z_h\} = \mathrm{Cl}^1 X$ where the z_i's are ordered such that $z_{2i} - x = x - z_{2i+1}$ for $i = 1, \dots, (h-1)/2$ and $\{t_1, \dots, t_{r'}\} = \mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X$. Then we can illustrate the graph of Φ_x as in Figure 4.

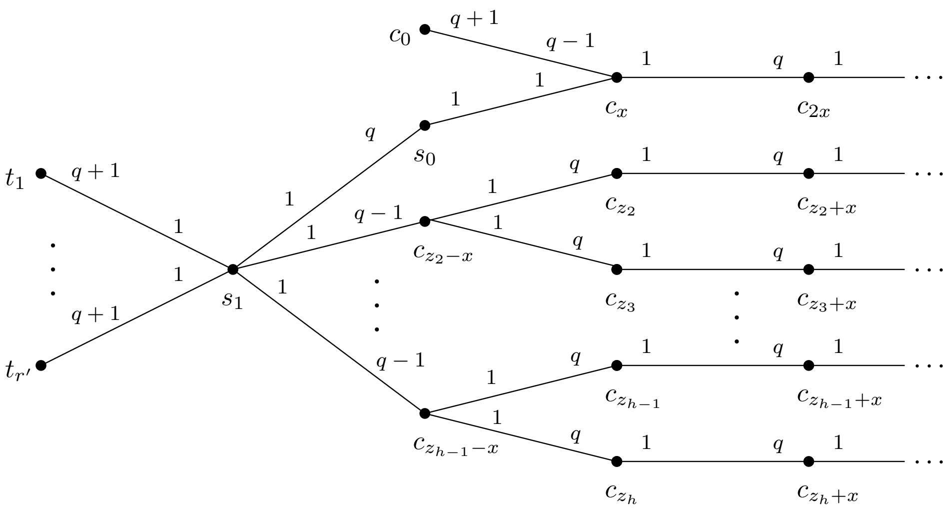


FIGURE 4. $\mathcal{G}_x$ for a degree one place x of an elliptic curve with odd class number

We give two examples for elliptic curves with even class number. Both examples are elliptic curves over $\mathbb{F}_3$ with class number 4, but with respective class group $\mathbb{Z}/4\mathbb{Z}$ and $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

4.3. **Example.** The first example is the elliptic curve X_5 over $\mathbb{F}_3$ defined by the Weierstrass equation $\underline{Y}^2 = \underline{X}^3 + \underline{X} + 2$, which has class group $\mathrm{Cl}^0 X_5 \simeq \mathbb{Z}/4\mathbb{Z}$. Let $X(\mathbb{F}_3) = \{x, y, z, z'\}$ such that $x - y$ is the element of order 2. The number of components is $h_2 = 2$, and $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}} X$ is given by s_0, $s_x = s_y$ and $s_z = s_{z'}$. The class group of $X_5' = X_5 \otimes \mathbb{F}_9$ is $\mathrm{Cl}^0 X_5' \simeq (\mathbb{Z}/4\mathbb{Z})^2$, thus $\mathrm{Cl}^0 X_5' \,/\, \mathrm{Cl}^0 X_5 \simeq \mathbb{Z}/4\mathbb{Z}$. Let $\{0, D, D', D''\}$ be representatives such that D is the divisor with $2D \in \mathrm{Cl}^0 X_5$. Then $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}} X_5$ contains the two elements t_D and $t_{D'} = t_{D''}$. We do not need to calculate the norm map $\mathrm{Cl}^0 X_5' \to \mathrm{Cl}^0 X_5$ as we can find out to which of t_D and $t_{D'}$ the vertices s_x and s_z are connected by the constraint that the weights around s_x and s_z, respectively, sum up to 4. The graph is illustrated in Figure 5.

4.4. **Example.** The second example X_6 over $\mathbb{F}_3$ is defined by the Weierstrass equation $\underline{Y}^2 = \underline{X}^3 + 2\underline{X}$, and has class group $\mathrm{Cl}^0 X_6 \simeq (\mathbb{Z}/2\mathbb{Z})^2 = \{x, y, z, w\}$. Here $h_2 = 4$, and s_x,

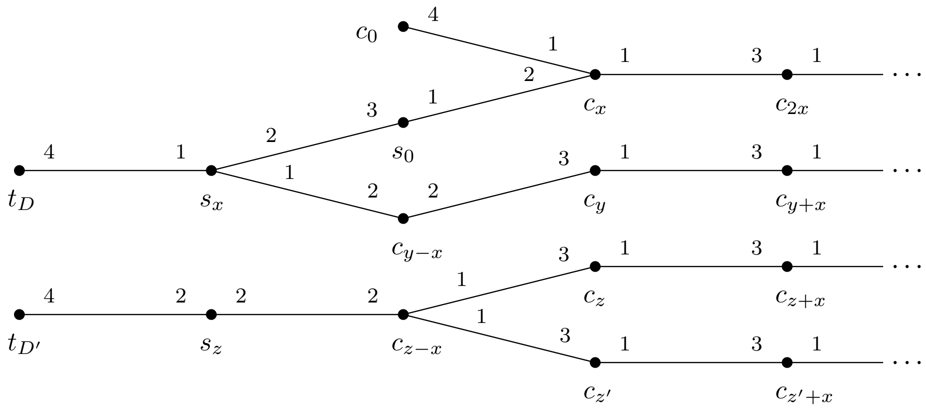


FIGURE 5. $\mathcal{G}_x$ for a degree one place x of the elliptic curve X_5

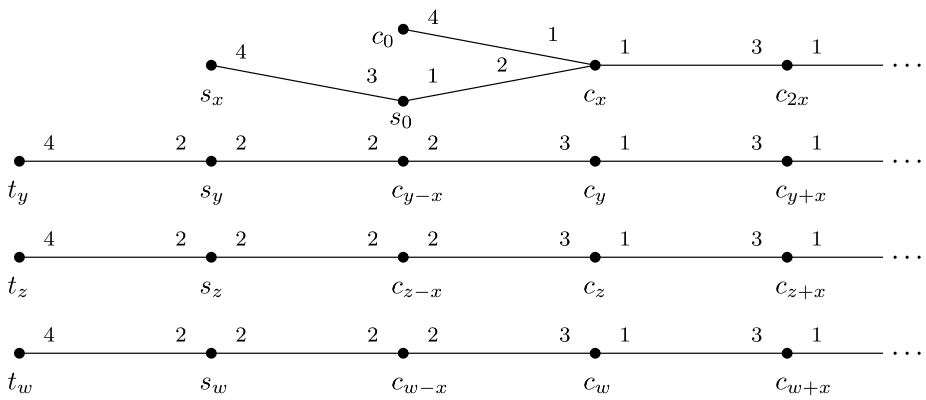


FIGURE 6. $\mathcal{G}_x$ for a degree one place x of the elliptic curve X_6

s_y, s_z and s_w are pairwise distinct vertices. For $X_6' = X_6 \otimes \mathbb{F}_9$, $\mathrm{Cl}^0\, X_6' \simeq (\mathbb{Z}/4\mathbb{Z})^2$, thus $\mathrm{Cl}^0\, X_6' \,/\, \mathrm{Cl}^0\, X_6 \simeq (\mathbb{Z}/2\mathbb{Z})^2$, which we represent by $\{0, D_1, D_2, D_3\}$, each of the D_i being of order 2. Again, by the constraint that weights around each vertex sum up to 4, we find that $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}}\, X_6$ contains three different traces of the line bundles corresponding to D_1, D_2 and D_3, which we denote by t_y, t_z and t_w, and which are connected to s_y, s_z and s_w, respectively. The graph is illustrated in Figure 6.

Part 2. **Applications to automorphic forms**

5. AUTOMORPHIC FORMS AS FUNCTIONS ON GRAPHS

In this section, we will review the notion of an automorphic form as a function on the vertex set $\mathbb{P}\mathrm{Bun}_2\, X$ of the graphs $\mathcal{G}_x$. These geometrically defined automorphic forms correspond to unramified automorphic forms for PGL_2 over F in the more common adelic language. The geometric viewpoint lets us extract explicit eigenvalue equations from the graphs $\mathcal{G}_x$, which we

will list at the end of this section. This allows us to calculate the space of cusp forms and the space of toroidal automorphic forms in the following sections.

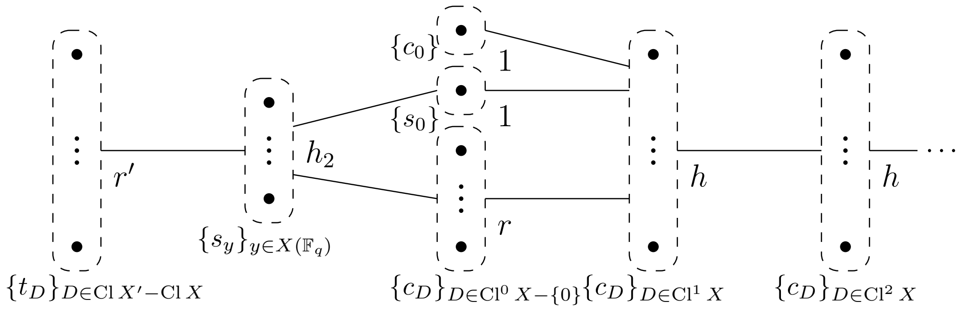


FIGURE 7. Certain subsets of $\mathrm{Vert}\ \mathcal{G}_x$ and their cardinalities

5.1. To begin with, we describe a useful picture of the graphs $\mathcal{G}_x$. Define the following numbers:

$$
\begin{aligned}
h\ &= \#\,\mathrm{Cl}^0\,X = \#\{c_D\}_{D\in\mathrm{Cl}^1\,X}, & h'\ &= \#(\mathrm{Cl}^0\,X'/\,\mathrm{Cl}^0\,X),\\
h_2 &= \#\,\mathrm{Cl}^0\,X[2] = \#\{s_y\}_{y\in X(\mathbb{F}_q)}, & h_2' &= \#(\mathrm{Cl}^0\,X'/\,\mathrm{Cl}^0\,X)[2],\\
r\ &= (h+h_2)/2-1 = \#\{c_D\}_{D\in\mathrm{Cl}^0\,X-\{0\}}, & r'\ &= (h'+h_2')/2-1 = \#\{t_D\}_{D\in\mathrm{Cl}\,X'-\mathrm{Cl}\,X}.
\end{aligned}
$$

The equality in the definition of h_2 follows from Theorem 2.4 and the equalities in definitions of h, r and r' follow from what we explained in paragraph 1.2. Figure 7 shows certain subsets of $\mathrm{Vert}\ \mathcal{G}_x$. Each dashed subset of $\mathrm{Vert}\ \mathcal{G}_x$ is defined by the set written underneath. The integer written to the right is its cardinality. A line between two dashed areas indicates that there is at least one edge in $\mathcal{G}_x$ between two vertices in the corresponding subsets.

5.2. **Lemma.** $h' \ = \ 2(q+1)-h.$

Proof. Fix a place x of degree 1 and consider $\mathcal{G}_x$. We count the weights around the h_2 vertices s_y, where y varies through $\mathrm{Cl}^1\,X$ modulo adding a class in $2\,\mathrm{Cl}^0\,X$. We know that the weights around each of the h_2 vertices add up to $q+1$. On the other hand, Theorem 3.1 tells us precisely which vertices occur as Φ_x-neighbours of the s_y's and with which weight. We count all weights around the s_y's:

- The vertex s_0 occurs with weight h_2.
- The vertex c_{z-x} occurs with weight h_2 if $z-x\in\mathrm{Cl}^0\,X-\{0\}$ and $z-x\neq x-z$.
- The vertex c_{z-x} occurs with weight $h_2/2$ if $z-x\in\mathrm{Cl}^0\,X-\{0\}$ and $z-x=x-z$.
- The vertex t_D occurs with weight h_2 if $D\in\mathrm{Cl}\,X'-\mathrm{Cl}\,X$ and $2D\notin\mathrm{Cl}\,X$.
- The vertex t_D occurs with weight $h_2/2$ if $D\in\mathrm{Cl}\,X'-\mathrm{Cl}\,X$ and $2D\in\mathrm{Cl}\,X$.

Since $c_{z-x}\ =\ c_{x-z}$, the sum of the weights of the c_{z-x}'s is $(h_2/2)(h-1)$. Since $t_D\ =\ t_{-D}$ and t_D depends only on the class of D modulo $\mathrm{Cl}\,X$, the sum of the weights of the t_D's is $(h_2/2)(h'-1)$. Adding up all these contributions gives

$$h_2(q+1)\ =\ h_2\ +\ (h_2/2)\,(h-1)\ +\ (h_2/2)\,(h'-1)\ ,$$

which implies the relation of the lemma. $\square$

5.3. **Remark.** This result can also be obtained from the equality $\zeta_{F'}(s) = \zeta_F(s) L_F(\chi, s)$, where $\chi = |\ |^{\pi \mathrm{i}/\ln q}$ is the Hecke character corresponding to F' by class field theory. These function can be written out explicitly as

$$\frac{q^2T^4 + (hh' - q^2 - 1)T^2 + 1}{(1 - T^2)(1 - q^2T^2)} \quad = \quad \frac{qT^2 + (h - q - 1)T + 1}{(1 - T)(1 - qT)} \cdot \frac{qT^2 - (h - q - 1)T + 1}{(1 + T)(1 + qT)} \ ,$$

where $T = q^{-s}$. Comparing the coefficients of the numerators of these rational functions in T yields an alternative proof of the lemma.

5.4. We review the notion of Hecke operators and automorphic forms in geometric language. Let $\mathcal{V}$ be the space of complex valued functions on the vertex set $\mathbb{P}\mathrm{Bun}_2 X$ of the graphs $\mathcal{G}_x$. The action of the *Hecke operators* Φ_x on $\mathcal{V}$ is as follows. Given a function $f \in \mathcal{V}$ and a vertex $v \in \mathbb{P}\mathrm{Bun}_2 X$, we define

$$\Phi_x(f)(v) \quad = \quad \sum_{(v,v',m) \,\in\, \mathrm{Edge}\ \mathcal{G}_x} m \cdot f(v').$$

The actions of Φ_x and Φ_y commute for every $x, y \in |X|$. The *Hecke algebra* is the free algebra $\mathcal{H} = \mathbb{C}[\Phi_x]_{x \in |X|}$ generated by the Hecke operators Φ_x. It acts on $\mathcal{V}$ by linear continuation of the action of the Φ_x.

A function $f \in \mathcal{V}$ is an *automorphic form* if f is contained in a finite-dimensional $\mathcal{H}$-invariant subspace of $\mathcal{V}$. We denote the space of automorphic forms by $\mathcal{A}$. Note that f is automorphic if and only if $\{\Phi_x^i(f) \mid i \geq 0\}$ is contained in a finite-dimensional subspace of $\mathcal{V}$ for one $x \in |X|$. Moreover, every Φ_x-eigenspace for a fixed place x has a basis that consists of simultaneous eigenfunctions for all $\Phi \in \mathcal{H}_K$. We call such a simultaneous eigenfunction an *$\mathcal{H}$-eigenfunction*.

5.5. **Remark.** An automorphic form as defined here corresponds to an unramified automorphic form for PGL_2 over the function field F and an element $\Phi \in \mathcal{H}$ corresponds to an unramified Hecke operators for PGL_2. For more details on this correspondence, see section 5 in [8]. Note that there is a digression in notation: all upper and lower indices K are suppressed in this paper. For instance, we write $\mathcal{A}$ for the space denoted by $\mathcal{A}^K$ and $\mathcal{H}$ for the algebra denoted by $\mathcal{H}_K$ in [8] and [9].

5.6. Let $f \in \mathcal{A}$ be an $\mathcal{H}$-eigenfunction and let λ_x be the eigenvalue for Φ_x where $x \in |X|$. We can evaluate the eigenvalue equation $\Phi_x(f) = \lambda_x f$ at each vertex v of $\mathcal{G}_x$ and for each place x of degree 1, which yields the following equations for the vertices in the nucleus. (Note that the expressions in the column on the left are labels, which will be used for the purpose of reference.)

$$
\begin{array}{lll}
(x,t_D) & \lambda_x f(t_D) = (q+1)f(s_{D+\sigma D+x}) & \text{for } D\in \operatorname{Cl}X' - \operatorname{Cl}X,\\
(x,s_0) & \lambda_x f(s_0) = qf(s_x)+f(c_x)\ , & \\
(x,c_0) & \lambda_x f(c_0) = (q+1)f(c_x)\ , & \\
(x,c_{z-x}) & \lambda_x f(c_{z-x}) = (q-1)f(s_z)+f(c_z)+f(c_{2x-z}) & \text{for } z\in X(\mathbb{F}_q)-\{x\},\\
(x,c_x) & \lambda_x f(c_x) = (q-1)f(s_0)+f(c_0)+f(c_{2x})\ , & \\
(x,c_z) & \lambda_x f(c_z) = qf(c_{z-x})+f(c_{z+x}) & \text{for } z\in X(\mathbb{F}_q)-\{x\},\\
(x,s_y) & \lambda_x f(s_y) = \alpha f(s_0) + (h_2/2)\sum\limits_{\substack{(z-x)\in \operatorname{Cl}^0 X\\ (z-x)\neq 0\\ (z-y)\in 2\operatorname{Cl}^0 X}} f(c_{z-x}) + (h_2/2)\sum\limits_{\substack{[D]\in \operatorname{Cl}X'/\operatorname{Cl}X\\ [D]\neq \operatorname{Cl}X\\ D-\sigma D+x-y\in 2\operatorname{Cl}^0 X}} f(t_D) &
\end{array}
$$

$$
\text{for } y\in X(\mathbb{F}_q)\text{, where } \alpha = \begin{cases} h_2 & \text{if } (y-x)\in 2\operatorname{Cl}^0 X,\\ 0 & \text{if } (y-x)\notin 2\operatorname{Cl}^0 X.\end{cases}
$$

If we add up all the eigenvalue equations evaluated in the vertices s_y, where we let y range over all of $X(\mathbb{F}_q) = \operatorname{Cl}^1 X$, then we obtain that

$$
(x,\textstyle\sum s_y) \qquad \sum_{y\in X(\mathbb{F}_q)} \lambda_y f(s_y) \;=\; hf(s_0) + (h/2)\sum_{\substack{(z-x)\in \operatorname{Cl}^0 X\\ (z-x)\neq 0}} f(c_{z-x}) + (h/2)\sum_{\substack{[D]\in \operatorname{Cl}X'/\operatorname{Cl}X\\ [D]\neq \operatorname{Cl}X}} f(t_D)\ .
$$

6. The space of cusp forms

In this section, we use our knowledge about the graphs $\mathcal{G}_x$ to investigate the space $\mathcal{A}_0$ of cusp forms. This means that we calculate its dimension and determine the (maximal) support of a cusp form. Further we employ a Hecke operator Φ_y for a place y of degree 2 to show that a cusp form that is a $\mathcal{H}$-eigenfunction does not vanish in c_0.

6.1. A *cusp form* is an automorphic form $f\in\mathcal{A}$ that satisfies the equation

$$
\sum_{\mathcal{M}\in \operatorname{Ext}^1(\mathcal{O}_X,\mathcal{O}_X)} \Phi(f)(\mathcal{M}) \quad = \quad 0
$$

for all $\Phi\in\mathcal{H}$ (cf. [4]). We denote the space of cusp forms by $\mathcal{A}_0$. The space of cusp forms admits a basis of $\mathcal{H}$-eigenfunctions, and is thus invariant under $\mathcal{H}$. The support of a cusp form is contained in the set of vertices v with $\delta(v)\leq 0$ (cf. [5] or [8, par. 9.3]).

Let h_2 and r' be as in the previous section. We obtain the following result.

6.2. **Theorem.**

(i) *The dimension of* $\mathcal{A}_0$ *is* $r'+1-h_2$.
(ii) *The support of* $f\in\mathcal{A}_0$ *is contained in* $\{t_D, s_0, c_0\}_{D\in \operatorname{Cl}X'-\operatorname{Cl}X}$.
(iii) *If* x *is a place of odd degree, then* $\Phi_x(f)=0$.

Proof. Let f be an $\mathcal{H}$-eigenform, which means, in particular, that f is not trivial, and λ_x be the eigenvalue for Φ_x where $x\in|X|$. We first show that $\lambda_x = 0$ if x is of degree 1.

Assume that $\lambda_x\neq 0$, then we conclude successively:

- $f(c_0) = 0$ by equation (x, c_0).
- $f(s_0) = 0$ by equation (x, c_x).
- $f(c_{z-x}) = 0$ for all places $z \neq x$ of degree 1 by equation (x, c_z).
- $f(s_y) = 0$ for all places y of degree 1 by equations (x, s_0) and (x, c_{z-x}).
- $f(t_D) = 0$ for all $D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$ by equation (x, t_D).

Thus f must be trivial, which contradicts our assumption on f. This shows that $\lambda_x = 0$ for all places x of degree 1. We make the following successive conclusions:

- $f(s_y) = 0$ for all places y of degree 1 by equation (x, t_D).
- $f(c_{z-x}) = 0$ for all places $z \neq x$ of degree 1 by equation (x, c_z).
- $f(c_0) + (q-1)f(s_0) = 0$ by equation (x, c_x).
- $\alpha f(s_0) + (h_2/2) \sum\limits_{\substack{[D]\in \mathrm{Cl}\, X'/\,\mathrm{Cl}\, X \\ [D]\neq \mathrm{Cl}\, X \\ D-\sigma D+x-y\in 2\,\mathrm{Cl}^0 X}} f(t_D) = 0$ for all places y of degree 1 by equation (x, s_y),

  where $\alpha = h_2$ if $(y - x) \in 2\,\mathrm{Cl}^0 X$ and $\alpha = 0$ otherwise.

This means that the support of f is contained in $\{t_D, s_0, c_0\}_{D\in \mathrm{Cl}\, X' - \mathrm{Cl}\, X}$, which proves (ii).

We have $h_2 + 1$ linearly independent equations for f as described by the last two lines of the above list. There are no further restrictions on the values of f given by the eigenvalue equations since equation (x, c_0) becomes trivial. Hence the dimension of the 0-eigenspace of Φ_x in $\mathcal{A}_0$ equals

$$\#\{t_D, s_0, c_0\}_{D\in \mathrm{Cl}\, X' - \mathrm{Cl}\, X} - (h_2 + 1) \;=\; (r' + 2) - (h_2 + 1) \;=\; r' + 1 - h_2\ .$$

Since there are no other eigenvalues for cusp forms, $\mathcal{A}_0$ equals the 0-eigenspace, which proves (i).

Assertion (iii) follows since the support of f contains only vertices v with $\delta(v)$ even and the parity of two Φ_x-neighbours is different if x is of odd degree (cf. [8, Lemma 8.2]). This implies that $\Phi_x(f) = 0$ for every place x of odd degree, which proves (iii). □

6.3. **Remark.** The dimension can be calculated by other methods, too. Once the vertices of the graphs $\mathcal{G}_x$ are determined, one can use theta series to calculate the dimension, cf. [11, Satz 3.3.2] and [5, Thm. 5.1].

6.4. **Proposition.** *If $f \in \mathcal{A}_0$ is an $\mathcal{H}$-eigenfunction, then $f(c_0) \neq 0$.*

Proof. Let $f \in \mathcal{A}_0$ be an $\mathcal{H}$-eigenfunction with Φ_x-eigenvalue λ_x such that $f(c_0) = 0$. We will deduce that f must be the zero function, which contradicts the assumption that f is an $\mathcal{H}$-eigenfunction. This will prove the lemma.

First we conclude from Theorem 6.2 and equation (x, c_x) that $f(s_0) = 0$. The only other vertices that are possibly contained in the support of f are of the form t_D for a $D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$. We fix an arbitrary $D \in \mathrm{Cl}\, X' - \mathrm{Cl}\, X$ for the rest of the proof and show that $F(t_D) = 0$.

Since $X'(\mathbb{F}_{q^2}) = \mathrm{Cl}^1 X'$ maps surjectively to $\mathrm{Cl}\, X'/\,\mathrm{Cl}\, X$, and t_D only depends on the class $[D] \in \mathrm{Cl}\, X'/\,\mathrm{Cl}\, X$, there is a $z \in X'(\mathbb{F}_{q^2})$ such that $t_D = t_z$. The covering $p : X' \to X$ maps z as well as its Galois conjugate σz to a place $y \in |X|$ of degree 2. As classes in $\mathrm{Cl}\, X'$, we have $y = z + \sigma z$.

In the following, we will investigate the graph of the Hecke operator Φ_y with the help of the graphs of the Hecke operators Φ_z and $\Phi_{\sigma z}$, which are defined over F'. Recall from [8, Lemma

6.5] that the map $p^* : \mathbb{P}\mathrm{Bun}_2\, X \to \mathbb{P}\mathrm{Bun}_2\, X'$ restricts to an injective map

$$p^* :\ \mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}}\, X \ \amalg\ \mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}}\, X \ \hookrightarrow\ \mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}}\, X' \ ,$$

and p^* maps $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}}\, X$ to $\mathbb{P}\mathrm{Bun}_2^{\mathrm{gi}}\, X'$. We will denote the elements in $\mathbb{P}\mathrm{Bun}_2^{\mathrm{dec}}\, X'$ by c'_D with $D \in \mathrm{Cl}\, X'$. Then we have in particular that $c'_0 = p^*(c_0)$, that $c'_{z+\sigma z} = p^*(c_y)$ and that $c'_{z-\sigma z} = p^*(t_z)$, and in each case, there is no other vertex in $\mathbb{P}\mathrm{Bun}_2\, X$ that is mapped to c'_0, $c'_{z+\sigma z}$, and $c'_{z-\sigma z}$, respectively.

Recall from paragraph 1.1 that $\mathcal{K}_y$ denotes the sheaf on X that is supported at y with stalk κ_y. If we denote by $\mathcal{K}_z$ and $\mathcal{K}_{\sigma z}$ the corresponding sheaves on X', we have that $p^*\mathcal{K}_y \simeq \mathcal{K}_z \oplus \mathcal{K}_{\sigma z}$.

Let $\mathcal{M}, \mathcal{M}' \in \mathrm{Bun}_2\, X$ fit into an exact sequence

$$0 \longrightarrow \mathcal{M}' \longrightarrow \mathcal{M} \longrightarrow \mathcal{K}_y \longrightarrow 0 \ .$$

Extension of constants is an exact functor, thus we obtain an exact sequence

$$0 \longrightarrow p^*\mathcal{M}' \longrightarrow p^*\mathcal{M} \longrightarrow \mathcal{K}_z \oplus \mathcal{K}_{\sigma z} \longrightarrow 0 \ ,$$

which splits into two exact sequences

$$0 \to \mathcal{M}'' \to p^*\mathcal{M} \to \mathcal{K}_z \to 0 \qquad \text{and} \qquad 0 \to p^*\mathcal{M}' \to \mathcal{M}'' \to \mathcal{K}_{\sigma z} \to 0 \ ,$$

where $\mathcal{M}'' \in \mathrm{Bun}_2\, X'$ is the kernel of $p^*\mathcal{M} \to \mathcal{K}_z$.

In the language of graphs, this means that for every edge

$$\underset{v}{\bullet} \xrightarrow{\ m\ } \underset{v'}{\bullet}$$

between vertices $v, v' \in \mathbb{P}\mathrm{Bun}_2\, X$ in $\mathrm{Edge}\, \mathcal{G}_y$, there are a vertex $v'' \in \mathbb{P}\mathrm{Bun}_2\, X'$, and edges

$$\underset{v}{\bullet} \xrightarrow{\ m'\ } \underset{v''}{\bullet} \qquad \text{and} \qquad \underset{v''}{\bullet} \xrightarrow{\ m''\ } \underset{v'}{\bullet}$$

in $\mathrm{Edge}\, \mathcal{G}_z$ and $\mathrm{Edge}\, \mathcal{G}_{\sigma z}$, respectively.

We apply this observation to find out all possibilities of Φ_y-neighbours of c_0. The only Φ_z-neighbour of c_0 is c_z, and since $z \neq \sigma z$, the $\Phi_{\sigma z}$-neighbours of c_z are $c_{z-\sigma z} = p^*(t_z)$ and $c_{z+\sigma z} = p^*(c_y)$. This means that the only possible Φ_y-neighbours of c_0 are t_z and c_y. The vertex c_y has multiplicity $q+1$ by [8, Thm. 8.5]. Thus the neighbour t_z has multiplicity $(q^2+1) - (q+1) = q^2 - q$. Hence $\mathcal{U}_y(c_0)$ can be illustrated as

$$\underset{t_z}{\bullet} \xleftarrow{\ q^2-q\ } \underset{c_0}{\bullet} \xrightarrow{\ q+1\ } \underset{c_y}{\bullet}$$

By the assumptions on f, it vanishes both at c_0 and at c_y. Thus the eigenvalue equation

$$\lambda_f(\Phi_y)\, f(c_0) \ = \ (q+1)\, f(c_y) + (q^2 - q)\, f(t_z)$$

implies that $f(t_D) = f(t_z) = 0$, which completes the proof. $\square$

7. The space of toroidal automorphic forms

In this section, we calculate the space of toroidal automorphic forms. We start with F'-toroidal automorphic forms, which have a particular nice geometric description, and prove in particular hat there are no non-trivial F'-toroidal cusp forms. From this result together with the investigation of the zeros of certain L-functions, we shall show that the space of toroidal automorphic forms is generated by a single Eisenstein series $E(\,\cdot\,,s)$ where $s+1/2$ is a zero of the zeta function of F—with the exception that the characteristic of F is 2 and the class number h of F equals $q+1$ where the space of toroidal automorphic forms might be 2-dimensional.

7.1. Let F' be the function field of X'. An automorphic form $f \in \mathcal{A}$ is *F'-toroidal* if

$$\sum_{[D]\in \mathrm{Cl}\, X'/\,\mathrm{Cl}\, X} \Phi(f)(t_D) \;=\; 0\;.$$

for all $\Phi \in \mathcal{H}$ (cf. [8, Thm. 10.8]). We denote the space of all F'-toroidal automorphic forms by $\mathcal{A}_{\mathrm{tor}}(F')$. This space is a $\mathcal{H}$-invariant subspace of $\mathcal{A}$, and it decomposes into a direct sum of the following three $\mathcal{H}$-invariant subspaces: the space $\mathcal{A}_{0,\mathrm{tor}}(F')$ of F'-toroidal cusp forms, the space $\mathcal{E}_{\mathrm{tor}}(F')$ that is generated by F'-toroidal derivatives of Eisenstein series and the space $\mathcal{R}_{\mathrm{tor}}(F')$ that is generated by toroidal derivatives of residual Eisenstein series (see sections 6 and 7 in [9] for the definition of the derivative of a (residual) Eisenstein series; we shall also give a brief description in paragraph 7.6).

We shall investigate these subspaces in the following. We begin with $\mathcal{A}_{0,\mathrm{tor}}(F')$. Since $t_D = c_0$ if $D \in \mathrm{Cl}\, X$, the defining equations for F'-toroidality contain in particular the equation

$$(T) \qquad f(c_0) \;+ \sum_{\substack{[D]\in \mathrm{Cl}\, X'/\,\mathrm{Cl}\, X\\ [D]\neq \mathrm{Cl}\, X}} f(t_D) \;=\; 0\;.$$

for the unit $\Phi = 1$ in $\mathcal{H}$.

7.2. **Theorem.** *Let $F' = \mathbb{F}_{q^2}F$ be the constant field extension of F. Then the space of unramified F'-toroidal cusp forms is trivial.*

Proof. Since the support of unramified cusp forms is contained in $\mathbb{P}\mathrm{Bun}_2^{\mathrm{tr}}\, X \cup \{s_0, c_0\}$ (Theorem 6.2), after multiplying by $2/h$, equation $(x, \sum s_y)$ simplifies to

$$0 \;=\; 2\,f(s_0) \;\; + \sum_{\substack{[D]\in \mathrm{Cl}\, X'/\,\mathrm{Cl}\, X\\ [D]\neq \mathrm{Cl}\, X}} f(t_D)\;.$$

Subtracting equation (T) from it yields

$$0 \;=\; 2\,f(s_0) \;-\; f(c_0)\;.$$

For cusp forms, equation (x, c_x) reads

$$0 \;=\; (q-1)f(s_0) + f(c_0)$$

and this implies that $f(c_0) = f(s_0) = 0$. Thus Proposition 6.4 implies that $\mathcal{A}_{0,\mathrm{tor}} = \{0\}$. □

7.3. **Remark.** A formula of Waldspurger ([14, Prop. 7]) for number fields together with certain non-vanishing results implies that a cusp form f which is an $\mathcal{H}$-eigenfunction is E-toroidal if and only if $L(\pi, 1/2)\cdot L(\pi\otimes\chi_E, 1/2) = 0$ where E is a quadratic field extension of F, π is

the cuspidal representation generated by f and χ_E is the Hecke character associated to E by class field theory (cf. section 6 in [3]). Conjecturally the corresponding fact is also true in the function field case; see [10] for partial results.

In our framework, $E = F'$ and $\chi_{F'} = |\ |^{\pi i/\ln q}$. The case of genus 0 follows trivially since there are no (unramified) cusp forms. For genus 1, both $L(\pi, s)$ and $L(\pi \otimes \chi_{F'}, s)$ equal $\frac{1}{(1-q^{-s})(1-q^{1-s})}$ and therefore they do not vanish in $s = 1/2$. On the other hand, there are no toroidal cusp forms by the previous theorem. Thus the case of genus 1 follows.

7.4. Before we proceed to determine the rest of $\mathcal{A}_{\rm tor}(F')$, we need some facts on the zeros of the zeta function $\zeta_{F'}$ of F'. The zeta function

$$\zeta_F(s) = \frac{qT^2 \,+\, \big(h \,-\, (q+1)\big)T \,+\, 1}{(1 \,-\, T)\,(1 \,-\, qT)}$$

of F satisfies the functional equation $\zeta_F(1-s) = \zeta_F(s)$. Here h is the class number and $T = q^{-s}$. Thus the zeros ζ_F come in pairs $\{s, 1-s\}$ (modulo $\frac{2\pi i}{\ln q}\mathbb{Z}$). Note that this set contains only one element if $s \equiv 1-s \pmod{\frac{2\pi i}{\ln q}\mathbb{Z}}$. We say that $\{s, 1-s\}$ is a pair of zeros of order n if s is a zero of order n in the case that $s \not\equiv 1-s \pmod{\frac{2\pi i}{\ln q}\mathbb{Z}}$ respective if n is a zero of order $2n$ in the case that $s \equiv 1-s \pmod{\frac{2\pi i}{\ln q}\mathbb{Z}}$. Note that in the latter case s is always a zero of even degree ([9, Lemma 12.1]).

Since the denominator in the above equation for $\zeta_F(s)$ is a polynomial in $T = q^{-s}$ of degree 2, we see that $\zeta_F(s)$ has only one pair of zeros (modulo $\frac{2\pi i}{\ln q}\mathbb{Z}$), which is of order 1. The same is true for the L-series

$$L(\chi_{F'}, s) = \frac{qT^2 \,-\, \big(h \,-\, (q+1)\big)T \,+\, 1}{(1 \,-\, T)\,(1 \,-\, qT)},$$

namely its pair of zeros is $\{s + \frac{\pi i}{\ln q}, 1 - s - \frac{\pi i}{\ln q}\}$ if $\{s, 1-s\}$ is the pair of zeros of $\zeta_F(s)$. Here $\chi_{F'} = |\ |^{\pi i/\ln q}$ is the unramified Hecke character associated to F' by class field theory. However, the zeta function $\zeta_{F'}(s) = \zeta_F(s) \cdot L(\chi_{F'}, s)$ of F', considered as a function of s modulo $\frac{2\pi i}{\ln q}\mathbb{Z}$, has either two different pairs of zeros of order 1 or one pair of zeros of order 2.

7.5. **Lemma.** *The following are equivalent.*

(i) $\zeta_{F'}$ *has a pair of zeros of order* 2.
(ii) $\frac{1}{2} + \frac{\pi i}{2\ln q}$ *is a zero of* ζ_F.
(iii) $h = q + 1$.

Proof. Let s be a zero of ζ_F. Then (i) holds if and only if $1 - s \equiv s + \frac{\pi i}{\ln q} \pmod{\frac{2\pi i}{\ln q}\mathbb{Z}}$, which is equivalent to (ii).

Put $s = \frac{1}{2} + \frac{\pi i}{2\ln q}$ and $T = q^{-s} = iq^{-1/2}$. Then

$$\zeta_F(s) \;=\; \frac{qi^2q^{-1} + \big(h-(q+1)\big)\,iq^{-1/2} + 1}{(1-iq^{-1/2})(1-iq^{1/2})} \;=\; \big(h-(q+1)\big)\,\underbrace{\frac{iq^{-1/2}}{(1-iq^{-1/2})(1-iq^{1/2})}}_{\neq 0}$$

is zero if and only if $h = q+1$, hence the equivalence of (ii) and (iii). $\square$

7.6. We proceed with determining the space $\mathcal{A}_{\rm tor}(F')$. We give a brief description of (residual) Eisenstein series and their derivatives. An *unramified Hecke character* is a character on the

divisor class group $\mathrm{Cl}\,X$. In particular, the principal characters $|D|^s = q^{-s\deg D}$ are unramified Hecke characters. Let χ be an unramified Hecke character. Define $\lambda_x(\chi) = q_x^{1/2}\big(\chi^{-1}(x) + \chi(x)\big)$ for all $x \in |X|$ and all unramified Hecke characters χ where we consider x as a prime divisor and $q_x = q^{\deg x}$. The automorphic form $E(\,\cdot\,,\chi)$ is characterised, up to scalar multiple, by the eigenvalue equations

$$\Phi_x\big(E(\,\cdot\,,\chi)\big) \quad = \quad \lambda_x(\chi)\, E(\,\cdot\,,\chi)$$

for all $x \in |X|$ (see [9, pars. 3.5 and 9.2] for more details). If $\chi \neq |\;|^{\pm 1}$, then $E(\,\cdot\,,\chi)$ is called a *Eisenstein series* and if $\chi = |\;|^{\pm 1}$, then $R(\,\cdot\,,\chi) = E(\,\cdot\,,\chi)$ is called a *residual Eisenstein series* or a *residuum of an Eisenstein series*. The functional equation for Eisenstein series is a linear relation between $E(\,\cdot\,,\chi)$ and $E(\,\cdot\,,\chi^{-1})$. Indeed, $\lambda_x(\chi) = \lambda_x(\chi^{-1})$ for all $x \in |X|$.

Define $\lambda_x^-(\chi) = q_x^{1/2}\big(\chi^{-1}(x) - \chi(x)\big)$ for all $x \in |X|$ and all unramified Hecke characters χ. If $\chi^2 \neq 1$ where $1 = |\;|^0$ is the trivial character, then there is a unique automorphic form $E^{(1)}(\,\cdot\,,\chi)$ that satisfies the generalised eigenvalue equations

$$\Phi_x\big(E^{(1)}(\,\cdot\,,\chi)\big) \quad = \quad \lambda_x(\chi)\, E^{(1)}(\,\cdot\,,\chi) \;+\; \ln q_x\, \lambda_x^-(\chi)\, E(\,\cdot\,,\chi)$$

for all $x \in |X|$ ([9, Lemmas 11.2, 11.3 and 11.7]). The functions $E(\,\cdot\,,\chi)$ and $E^{(1)}(\,\cdot\,,\chi)$ span a 2-dimensional $\mathcal{H}$-invariant subspace of $\mathcal{A}$ ([9, Props. 11.4 and 11.8]). If $\chi^2 = 1$, then there is a unique automorphic form $E^{(2)}(\,\cdot\,,\chi)$ that satisfies the generalised eigenvalue equations

$$\Phi_x\big(E^{(2)}(\,\cdot\,,\chi)\big) \quad = \quad \lambda_x(\chi)\, E^{(2)}(\,\cdot\,,\chi) \;+\; (\ln q_x)^2\, \lambda_x(\chi)\, E(\,\cdot\,,\chi)$$

for all $x \in |X|$ ([9, Lemmas 11.2 and 11.3]). The functions $E(\,\cdot\,,\chi)$ and $E^{(2)}(\,\cdot\,,\chi)$ span a 2-dimensional $\mathcal{H}$-invariant subspace of $\mathcal{A}$ ([9, Prop. 11.6]).

Let h_2 be the number of 2-torsion elements in the class group. Then we can describe the space of F'-toroidal automorphic forms as follows.

7.7. **Theorem.** *Let $s + 1/2$ be a zero of ζ_F and W the set of all unramified Hecke characters $\chi = \omega\,|\;|^{1/2}$ such that $\omega^2 = 1$, but $\omega|_{\mathrm{Cl}^0 X} \neq 1$. If $h \neq q+1$, then $\mathcal{A}_{\mathrm{tor}}(F')$ is generated by*

$$\Big\{\, E(\,\cdot\,,|\;|^s),\; E(\,\cdot\,,|\;|^{s+\pi\mathrm{i}/\ln q}),\; R(\,\cdot\,,\chi) \,\Big\}_{\chi\in W}$$

and if $h = q+1$, then $\mathcal{A}_{\mathrm{tor}}(F')$ is generated by

$$\Big\{\, E(\,\cdot\,,|\;|^s),\; E^{(1)}(\,\cdot\,,|\;|^s),\; R(\,\cdot\,,\chi) \,\Big\}_{\chi\in W}\,.$$

In particular, $\dim \mathcal{A}_{\mathrm{tor}}(F') = 2h_2$.

Proof. Since $\mathcal{A}_{\mathrm{tor}}(F') = \mathcal{A}_{0,\mathrm{tor}}(F') \oplus \mathcal{E}_{\mathrm{tor}}(F') \oplus \mathcal{R}_{\mathrm{tor}}(F')$, we can investigate the three summands separately. By Theorem 7.2, we have $\mathcal{A}_{0,\mathrm{tor}}(F') = 0$. By [9, Thm. 7.6], we have $\mathcal{R}_{\mathrm{tor}}(F') = \{R(\,\cdot\,,\chi)\}_{\chi\in W}$. By definition, $\chi = \omega\,|\;|^{1/2} \in W$ if and only if ω is an unramified Hecke character that factors through $\mathrm{Cl}\,X \,/\, 2\,\mathrm{Cl}\,X$ and that is nontrivial if restricted to $\mathrm{Cl}^0\,F$. The group $\mathrm{Cl}\,X \,/\, 2\,\mathrm{Cl}\,X$ is a group of order $2h_2$, and its character group is of the same order. There are two quadratic characters such that $\omega|_{\mathrm{Cl}^0 X}$ is trivial, namely, the trivial character and $|\;|^{\pi\mathrm{i}/\ln q}$. Consequently, the cardinality of W is $2h_2 - 2$.

By [9, Thm. 4.3], an Eisenstein series $E(\,\cdot\,,\chi)$ is F'-toroidal if and only if $L(\chi,s)L(\chi\chi_{F'},s)$ vanishes in $s=1/2$. Since this happens only for principal characters $\chi=|\;|^s$, we may reformulate this condition as follows: $E(\,\cdot\,,|\;|^s)$ is F'-toroidal if and only if $\zeta_F(1/2+s)L(\chi_{F'},1/2+s)=0$. If $h\neq q+1$, then $\zeta_F(1/2+s)L(\chi_{F'},1/2+s)$ has two pairs of zeros of order 1 by Lemma 7.5. By [9, Thm. 6.2], the two Eisenstein series in the theorem are F'-toroidal and by [9, Thm. 4.3 (ii)], $\mathcal{E}_{\mathrm{tor}}(F')$ is generated by these two linear independent functions. If $h=q+1$, then $\{1/2+\frac{\pi\mathrm{i}}{2\ln q},1/2-\frac{\pi\mathrm{i}}{2\ln q}\}$ is a pair of zeros of order 2 by Lemma 7.5. Note that $\chi=|\;|^{\frac{\pi\mathrm{i}}{2\ln q}}$ is not of order 2, but of order 4. Thus by [9, Thms. 4.3 and 6.2], $E(\,\cdot\,,|\;|^s)$ and $E^{(1)}(\,\cdot\,,|\;|^s)$ form a basis of $\mathcal{E}_{\mathrm{tor}}(F')$ where $s=\frac{\pi\mathrm{i}}{2\ln q}$ is a zero of $\zeta_F(1/2+s)$.

The dimension $\mathcal{A}_{\mathrm{tor}}(F')$ is consequently $0+2+(2h_2-2)=2h_2$, which completes the proof. □

7.8. **Remark.** An alternative proof of the above theorem can be accomplished by considering eigenvalue equations, similar to the proof of Theorem 6.2. This is done in [7, section 8.4]. The interesting aspect of the calculations with the eigenvalue equations is that one obtains the equality $\lambda_x^2=(q+1-h)^2$ for the Φ_x-eigenvalues λ_x of the Eisenstein series $E(\,\cdot\,,\chi)$ in $\mathcal{A}_{\mathrm{tor}}(F')$ if x is of degree 1. The estimation $0<h<2q+2$ coming from the embedding of X into $\mathbb{P}^2$ yields that $\mathcal{A}_{\mathrm{tor}}(F')$ is unitarizable. To prove that $\mathcal{A}_{\mathrm{tor}}(F')$ is a tempered representation, which would imply the Riemann hypothesis for X (cf. [9, Thm. 9.4]), one needs the estimate $q+1-2q^{1/2}\leq h\leq q+1+2q^{1/2}$. For more details, cf. [7, par. 8.4.7].

The dependence of λ_x on h should come as no surprise, since the class number h has an important influence on the shape of the graphs $\mathcal{G}_x$. For a given elliptic curve with known class number, it is however possible to proof the Riemann hypothesis via these methods. Note that in this proof, we do not make use of the fact that the zeta function is a rational function. This leaves some hope that these methods can say something about zeta functions of number fields.

7.9. In the rest of this paper, we will determine the space of toroidal automorphic forms for an elliptic curve X. This will be done based on theorems of [9] and calculations with L-series.

There is, more general, a definition of an E-toroidal automorphic form for every separable quadratic algebra extension of F, which is either a separable quadratic field extension of F or isomorphic to $F\oplus F$. If $\mathcal{A}_{\mathrm{tor}}(E)$ denotes the space of (unramified) E-toroidal automorphic forms, then the space of *(unramified) toroidal automorphic forms* is the intersection $\mathcal{A}_{\mathrm{tor}}=\bigcap\mathcal{A}_{\mathrm{tor}}(E)$ where E ranges over all separable quadratic algebra extensions of F. We shall not recall these definitions, which can be found in [9, section 2].

Important for the following conclusions is the connection of E-toroidal automorphic Eisenstein series and zeros of L-series. Namely, let E be a separable quadratic algebra extension of F. Let χ_E be the Hecke character that is associated to E by class field theory. In particular, χ_E is the trivial character for $E\simeq F\oplus F$ and $\chi_{F'}=|\;|^{\pi\mathrm{i}/\ln q}$ for the function field F' of X'. Then an Eisenstein series $E(\,\cdot\,,\chi)$ is E-toroidal if and only if $L(\chi,1/2)L(\chi\chi_E,1/2)=0$ (cf. [9, Cors. 4.4 and 5.6]). For zeros of higher order, also derivatives of Eisenstein series are toroidal. The precise statement can be found in [9, Thm. 6.2]. Let $\mathcal{E}_{\mathrm{tor}}(E)$ denote the space that is generated by all E-toroidal derivatives of Eisenstein series.

We can draw some first conclusions. Since $\mathcal{A}_{\mathrm{tor}}\subset\mathcal{A}_{\mathrm{tor}}(F')$, there are no toroidal automorphic cusp forms. By [9, Thm. 7.7], there are no toroidal residues of Eisenstein series. So we are left to determine the space of toroidal (derivatives of) Eisenstein series, which contains

at most the 2-dimensional space $\mathcal{E}_{\rm tor}(F')$ as determined in Theorem 7.7 and at least the Eisenstein series $E(\chi,1/2)$ for which $L(\chi,1/2)=0$, which span an 1-dimensional subspace of $\mathcal{E}_{\rm tor}(F')$. We will show in the following that $\mathcal{A}_{\rm tor}$ equals this 1-dimensional subspace when the characteristic of F is odd or $h\neq q+1$.

7.10. **Proposition.** *Let $s+1/2$ be a zero of ζ_F. Then $\mathcal{E}_{\rm tor}(F\oplus F)$ is 2-dimensional. If $s\not\equiv 0 \pmod{\frac{\pi \mathrm{i}}{\ln q}\mathbb{Z}}$, then it is generated by*

$$\left\{\, E(\,\cdot\,,|\,|^s),\ E^{(1)}(\,\cdot\,,|\,|^s)\,\right\}\,,$$

and if $s\equiv 0 \pmod{\frac{\pi \mathrm{i}}{\ln q}\mathbb{Z}}$, then it is generated by

$$\left\{\, E(\,\cdot\,,|\,|^s),\ E^{(2)}(\,\cdot\,,|\,|^s)\,\right\}\,.$$

Proof. The Eisenstein series $E(\,\cdot\,,\chi)$ is $F\oplus F$-toroidal if and only if $L(\chi,1/2)\cdot L(\chi,1/2)=0$. The only pair of zeros of $\big(L(\chi,1/2)\big)^2$ is $\{|\,|^s,|\,|^{-s}\}$ and it is of order 2. Fix $\chi=|\,|^s$ such that $L(\chi,1/2)=\zeta_F(s+1/2)=0$.

If $s\not\equiv 0 \pmod{\frac{\pi \mathrm{i}}{\ln q}\mathbb{Z}}$, which is the case when $|\,|^s\neq|\,|^{-s}$, then $\big(\zeta_F(s+1/2)\big)^2$ vanishes in s to order 2. In this case, the space of $F\oplus F$-toroidal derivatives of Eisenstein series is spanned by $E(\,\cdot\,,|\,|^s)$, $E^{(1)}(\,\cdot\,,|\,|^s)$, $E(\,\cdot\,,|\,|^{-s})$ and $E^{(1)}(\,\cdot\,,|\,|^{-s})$ (cf. [9, Thm. 6.2]). By [9, Thm. 11.10], $E(\,\cdot\,,|\,|^s)$ and $E^{(1)}(\,\cdot\,,|\,|^s)$ form a basis for this space.

If $s\equiv 0 \pmod{\frac{\pi \mathrm{i}}{\ln q}\mathbb{Z}}$, which is the case when $|\,|^s=|\,|^{-s}$, then $\big(\zeta_F(s+1/2)\big)^2$ vanishes in s to order 4. In this case, the space of $F\oplus F$-toroidal derivatives of Eisenstein series is spanned by $E(\,\cdot\,,|\,|^s)$, $E^{(1)}(\,\cdot\,,|\,|^s)$, $E^{(2)}(\,\cdot\,,|\,|^s)$ and $E^{(3)}(\,\cdot\,,|\,|^s)$ (cf. [9, Thm. 6.2]). By [9, Thm. 11.10], $E(\,\cdot\,,|\,|^s)$ and $E^{(2)}(\,\cdot\,,|\,|^s)$ form a basis for this space. This completes the proof. □

7.11. **Proposition.** *Assume that F has a separable geometric quadratic unramified field extension E. Let χ_E be the corresponding Hecke character. Let $s+1/2$ be a zero of ζ_F. Then $\mathcal{E}_{\rm tor}(E)$ is 2-dimensional and generated by*

$$\left\{\, E(\,\cdot\,,|\,|^s),\ E(\,\cdot\,,\chi_E\,|\,|^s)\,\right\}\,.$$

Proof. As explained above, $E(\,\cdot\,,\chi)$ is E-toroidal if and only if χ is a zero of the product $L(\chi,1/2)L(\chi\chi_E,1/2)$. Note that a separable quadratic unramified field extension is geometric if and only if the corresponding Hecke character χ_E restricts to a non-trivial character on the class group $\mathrm{Cl}^0 X$. Therefore $|\,|^s$ cannot be the inverse of $\chi_E\,|\,|^s$. Consequently, the only (unramified) pair of zeros of $L(\chi,1/2)$ is $\{|\,|^s,|\,|^{-s}\}$ and it is of order 1. The only pair of zeros of $L(\chi\chi_E,1/2)$ is $\{\chi_E\,|\,|^s,\chi_E^{-1}\,|\,|^{-s}\}$ and it is of order 1. This proves the proposition. □

7.12. **Theorem.** *Let F be an elliptic function field with class number h and constants $\mathbb{F}_q$. Let $s+1/2$ be a zero of ζ_F.*

(i) *If either the characteristic of F is odd or $h\neq q+1$, then $\mathcal{A}_{\rm tor}$ is 1-dimensional and spanned by the Eisenstein series $E(\,\cdot\,,|\,|^s)$.*
(ii) *If the characteristic of F is 2 and $h=q+1$, then $\mathcal{A}_{\rm tor}$ is either 1-dimensional and spanned by $E(\,\cdot\,,|\,|^s)$ or 2-dimensional and spanned by $\big\{E(\,\cdot\,,|\,|^s),\ E^{(1)}(\,\cdot\,,|\,|^s)\big\}$.*

Proof. As explained in paragraph 7.9, $\mathcal{A}_{\rm tor}$ contains $E(\,\cdot\,,|\,|^s)$ and is contained in the 2-dimensional space $\mathcal{E}_{\rm tor}(F')$ generated by $E(\,\cdot\,,|\,|^s)$ and $E(\,\cdot\,,|\,|^{s+\pi \mathrm{i}/\ln q})$ if $h\neq q+1$, respective,

$E(\,\cdot\,, |\,|^s)$ and $E^{(1)}(\,\cdot\,, |\,|^s)$ if $h = q+1$ (as described in Theorem 7.7). It suffices to show that $E(\,\cdot\,, |\,|^{s+\pi i/\ln q})$ respective $E^{(1)}(\,\cdot\,, |\,|^s)$ is not toroidal if the characteristic of F is odd or $h \neq q+1$ to prove the theorem.

If $h \neq q+1$, then $\zeta_{F'}$ has simple zeros by Lemma 7.5. Hence the intersection of $\mathcal{E}_{\mathrm{tor}}(F')$ (as described in Proposition 7.7) with $\mathcal{E}_{\mathrm{tor}}(F \oplus F)$ (as described in Proposition 7.10) is 1-dimensional and spanned by $E(\,\cdot\,, |\,|^s)$.

If the characteristic is odd, then the result follows from [9, Thm. 8.2]. We can also deduce it easily from the preceding as follows. We can assume that $h = q+1$, so h is even. There is thus a separable geometric quadratic unramified field extension E/F that corresponds to a nontrivial character χ_E on the class group $\mathrm{Cl}^0 X$. Thus the intersection of $\mathcal{A}_{\mathrm{tor}}(F')$ with $\mathcal{E}_{\mathrm{tor}}(E)$ (as described in Proposition 7.11) is 1-dimensional and spanned by $E(\,\cdot\,, |\,|^s)$. □

7.13. **Corollary.** *Let F be an elliptic function field with constant field $\mathbb{F}_q$ and class number h. If either the characteristic of F is not 2 or $h \neq q+1$, then there is for every unramified Hecke character χ and for every $s \in \mathbb{C}$ a quadratic character $\omega \in \Xi$ such that $L(\chi\omega, s) \neq 0$.* □

7.14. **Remark.** The proof of the last theorem depends on many results from the theory for toroidal automorphic forms as developed in [8] and [9]. In the particular case that the class number is 1, however, it is possible to deduce the theorem comparatively quickly by the method described in the introduction (cf. [2]).

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